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On embeddings of homogeneous spaces with small boundary [☆]

Ivan V. Arzhantsev ^a, Jürgen Hausen ^{b,*}

^a *Department of Higher Algebra, Faculty of Mechanics and Mathematics, Moscow State Lomonosov University,
Vorobievsky Gory, GSP-2, Moscow, 119992, Russia*

^b *Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany*

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Abstract

We study equivariant embeddings with small boundary of a given homogeneous space G/H , where G is a connected linear algebraic group with trivial Picard group and only trivial characters, and $H \subset G$ is an extension of a connected Grosshans subgroup by a torus. Under certain maximality conditions, like completeness, we obtain finiteness of the number of isomorphism classes of such embeddings, and we provide a combinatorial description of the embeddings and their morphisms. The latter allows a systematic treatment of examples and basic statements on the geometry of the equivariant embeddings of a given homogeneous space G/H .

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* Corresponding author.

E-mail addresses: arjantse@mccme.ru (I.V. Arzhantsev), hausen@mail.mathematik.uni-tuebingen.de, hausen@everest.mathematik.uni-tuebingen.de (J. Hausen).

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0. Introduction

Homogeneous spaces G/H and their equivariant (open) embeddings $G/H \subset X$ are of central interest in various fields of mathematics. In the setting of algebraic geometry, there is a general approach to embeddings of homogeneous spaces due to Luna and Vust [21]. However, this approach preferably applies to the case of small complexity, and even then, due to its generality, it is a deep and complicated theory, compare [20] and [28]. In the present article, we concentrate on a (rather) special class of G/H -embeddings, and for these we provide a simple alternative approach, based on combinatorial methods in Geometric Invariant Theory.

More precisely, let G be a connected linear algebraic group over an algebraically closed field \mathbb{K} of characteristic zero, and assume that G has trivial Picard group and admits only trivial characters (e.g., is semisimple and simply connected). We consider subgroups $H \subset G$, which are “Grosshans extensions” in the sense that H is an extension of a connected Grosshans subgroup $H_1 \subset G$ by some torus $T \subset G$; recall from [15] that $H_1 \subset G$ is a Grosshans subgroup if and only if G/H_1 is quasiasfine with a finitely generated algebra of global functions.

Given such a Grosshans extension $H \subset G$, we investigate “small” equivariant embeddings $G/H \subset X$, where X is a normal variety, and small means that the boundary $X \setminus (G/H)$ is small, i.e., of codimension at least two in X . Here are some simple examples.

Example. For the special linear group $G := \mathrm{SL}(3, \mathbb{K})$, consider the connected Grosshans subgroup

$$H_1 := \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}; a, b \in \mathbb{K} \right\}.$$

Then the following G -varieties are small equivariant G/H -embeddings with a Grosshans extension $H \subset G$:

- (i) The product of projective spaces $\mathbb{P}(\mathbb{K}^3) \times \mathbb{P}(\mathbb{K}^3)$ with the diagonal G -action and the subgroup

$$H := \left\{ \begin{bmatrix} t_1 & 0 & a \\ 0 & t_2 & b \\ 0 & 0 & t_1^{-1}t_2^{-1} \end{bmatrix}; t_1, t_2 \in \mathbb{K}^*, a, b \in \mathbb{K} \right\}.$$

- (ii) The projective space $\mathbb{P}(\mathbb{K}^3 \oplus \mathbb{K}^3)$ with G -action induced from the diagonal G -action on $\mathbb{K}^3 \oplus \mathbb{K}^3$ and the subgroup

$$H := \left\{ \begin{bmatrix} t & 0 & a \\ 0 & t & b \\ 0 & 0 & t^{-2} \end{bmatrix}; t \in \mathbb{K}^*, a, b \in \mathbb{K} \right\}.$$

Note that in the first of these examples, the resulting homogeneous space is spherical, i.e., of complexity zero, whereas in the second case it is of complexity one, and we have infinitely many $\mathrm{SL}(3)$ -orbits. However, also in the first setting, the construction may be generalized to higher dimensions, see Proposition 4.5, and then it produces $\mathrm{SL}(n)/H$ -embeddings of arbitrary high complexity, with $H \subset \mathrm{SL}(n)$ still being a Grosshans extension.

A first result of this paper is the following finiteness statement concerning isomorphism classes, see Theorem 3.7 for the full statement.

Theorem. *Let G be a connected linear algebraic group with trivial Picard group and only trivial characters. Then, for a fixed Grosshans extension $H \subset G$, there are only finitely many isomorphism classes of complete small equivariant G/H -embeddings.*

Our main aim is to provide a combinatorial description of the possible small equivariant G/H -embeddings X and their morphisms for a fixed Grosshans extension $H \subset G$. This can be done under a maximality assumption: the variety X should be A_2 -maximal, that means that any two points of X admit a common affine neighborhood in X , and for every open embedding $X \subset X'$ into a variety X' with the same property such that $X' \setminus X$ is of codimension at least two in X' , we have $X = X'$. Examples of A_2 -maximal varieties are affine and projective ones, but there exist definitely more of them.

Our approach is based on ideas of [3], which we redevelop and enhance here in a more geometric setting, see Sections 1 and 2. We consider the canonical action of the torus $T = H/H_1$ on the homogeneous space G/H_1 . This action extends to the affine closure

$$Z := \mathrm{Spec}(\mathcal{O}(G)^{H_1}).$$

The key observation is that every small equivariant G/H -embedding X occurs as a good quotient space $U // T$ of a T -invariant open subset $U \subset Z$; compare also [17] for this point of view. Let us briefly see, what we obtain for the examples discussed before.

Example (continued). For $G := \mathrm{SL}(3, \mathbb{K})$, and the Grosshans subgroup $H_1 \subset G$ given as before, we have

$$Z = \mathrm{Spec}(\mathcal{O}(G)^{H_1}) \cong \mathbb{K}^3 \oplus \mathbb{K}^3.$$

Moreover, the open subsets $U \subset Z$ over (i) the product $\mathbb{P}(\mathbb{K}^3) \times \mathbb{P}(\mathbb{K}^3)$, and (ii) the projective space $\mathbb{P}(\mathbb{K}^3 \oplus \mathbb{K}^3)$ are given by

- (i) $U = \{(v_1, v_2); v_1 \neq 0 \neq v_2\}$,
- (ii) $U = \{(v_1, v_2); v_1 \neq 0 \text{ or } v_2 \neq 0\}$.

According to the key observation, our task is now a matter of Geometric Invariant Theory: find an appropriate description of the open T -invariant sets $U \subset Z$ admitting a good quotient $U \rightarrow U // T$ with an A_2 -maximal quotient space. Generalizing the description of projective quotients given in [4, Section 2], we present in Section 1 a description in terms of “orbit cones” $\omega(z)$, where $z \in Z$, which live in the rational character space $\mathbb{X}_{\mathbb{Q}}(T)$.

Let us explain this. For each $z \in Z$, define $\omega(z)$ to be the (convex, polyhedral) cone generated by the weights $\chi \in \mathbb{X}(T)$ admitting a semi-invariant $f \in \mathcal{O}(Z)$ with $f(z) \neq 0$. To any open subset $U \subset Z$ we associate the collection Ψ of the orbit cones $\omega(z)$, where $T \cdot z$ is closed in U . It turns out that for the sets $U \subset Z$ with an A_2 -maximal good quotient space we obtain precisely the 2-maximal collections Ψ , i.e., for any two cones of Ψ their relative interiors overlap, and Ψ is maximal with this property.

Among the open subsets $U \subset Z$ with an A_2 -maximal good quotient space, the small equivariant G/H -embeddings correspond to interior 2-maximal collections Ψ , i.e., those containing the generic orbit cone. Moreover, using the description of projective quotients in terms of the GIT-fan as provided in [4], it is easy to figure out the projective small equivariant embeddings. These observations are summarized in our second main result as follows, see Theorem 3.10.

Theorem. *Let G be a connected linear algebraic group with trivial Picard group and only trivial characters, and let $H \subset G$ be a Grosshans extension. Then there is an equivalence of categories:*

$$\left\{ \begin{array}{l} \text{interior 2-maximal collections} \\ \text{of orbit cones} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} A_2\text{-maximal small equivariant} \\ G/H\text{-embeddings} \end{array} \right\}.$$

If moreover $\mathcal{O}(G/H) = \mathbb{K}$ holds, then we have in addition the following equivalence of categories:

$$\{\text{interior GIT-cones}\} \rightarrow \left\{ \begin{array}{l} \text{projective small equivariant} \\ G/H\text{-embeddings} \end{array} \right\}.$$

Strictly speaking, an equivalence of categories needs a notion of morphisms on both sides. On the left-hand side, a morphism is a certain “face relation,” see Section 2 for the precise definition. On the right-hand side, we have, as usual, the equivariant base point preserving morphisms, see Section 3.

Example (continued). Let $G := \mathrm{SL}(3, \mathbb{K})$ and $H_1 \subset G$ be as before. Then, in the setting of (i), the torus $T = H/H_1$ is of dimension two, and in $\mathbb{X}(T) = \mathbb{Z}^2$, the orbit cones of the T -action on $Z = \mathbb{K}^3 \oplus \mathbb{K}^3$ are the generic orbit cone

$$\omega(Z) = \mathrm{cone}(e_1, e_2) \subset \mathbb{Q}^2$$

and its faces. In particular, $\mathbb{P}(\mathbb{K}^3) \times \mathbb{P}(\mathbb{K}^3)$ is the only A_2 -maximal small equivariant G/H -embedding. A little more variation takes place, if one considers a smaller torus extension of the Grosshans subgroup $H_1 \subset G$, for example:

$$H' := \left\{ \begin{bmatrix} t & 0 & a \\ 0 & t^{-1} & b \\ 0 & 0 & 1 \end{bmatrix}; t \in \mathbb{K}^*, a, b \in \mathbb{K} \right\}.$$

The action of $T' = H'/H_1$ on Z has besides the generic orbit cone $\mathbb{Q} = \mathbb{X}_{\mathbb{Q}}(T')$ the two rays $\mathbb{Q}_{\leq 0}$ and $\mathbb{Q}_{\geq 0}$, and the zero cone $\{0\}$ as its orbit cones. The latter three form the GIT-fan, and the corresponding G/H' -embeddings are the only A_2 -maximal ones; see Example 4.8 for more details.

As mentioned before, the construction of examples seen so far is put in Section 4 into a general framework. We provide several general constructions of spherical and nonspherical examples. For simple groups G , we give in Section 4 a classification of the small equivariant G/H -embeddings that additionally admit the structure of a toric variety, see Proposition 4.7. Finally, we also construct some nontoric examples, see Proposition 4.9.

In the last section, we study geometric properties of small equivariant G/H -embeddings. The basic observation is that we may apply the language of bunched rings developed in [3]. For example, existence of projective small embeddings with at most \mathbb{Q} -factorial singularities immediately drops out, see Corollary 5.5 and compare [9, Théorème 1]; or one may produce nonprojective complete small G/H -embeddings. Moreover, we can easily construct examples of homogeneous spaces G/H admitting equivariant completions with small boundary but no smooth ones, see Example 5.8 and compare [10] for a discussion of such phenomena.

1. Quotients of affine torus actions

Given an algebraic variety with a reductive group action, it is one of the basic tasks of Geometric Invariant Theory to describe all invariant open subsets admitting a so-called “good quotient.” The “variation of quotients” problem is to understand the relations between these good quotients. For reductive group actions on projective varieties, there is a satisfactory picture, concerning good quotients that arise via Mumford’s method [22] from linearized ample bundles, see [11, 13, 25, 27].

In [4, Section 2], the problem for quasiprojective quotient spaces of the action of a torus T on a factorial affine variety Z was considered, and an elementary construction of the describing GIT-fan was given. Here we ask more generally for open subsets $U \subset Z$ that admit a quotient space, which is embeddable into a toric variety. Our result generalizes a similar result obtained in [7] for linear representations of tori.

We first fix our notation. By a lattice we mean a finitely generated free abelian group. For any lattice K , we denote by $K_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} K$ the associated rational vector space. The word cone always stands for a convex, polyhedral cone in a rational vector space. For a cone σ , we denote its relative interior by σ° and we write $\tau \preceq \sigma$ if τ is a face of σ .

In the sequel, \mathbb{K} is an algebraically closed field, and, for the sake of rigorous references, we suppose \mathbb{K} to be characteristic zero (though we expect our results to hold as well in positive characteristics). Moreover, K is a lattice, $T := \operatorname{Spec}(\mathbb{K}[K])$ is the corresponding algebraic torus, and we fix an K -graded integral affine \mathbb{K} -algebra

$$R = \bigoplus_{u \in K} R_u.$$

Recall that the K -grading of R defines a T -action on the corresponding affine variety $Z := \operatorname{Spec}(R)$ such that the homogeneous $f \in R_u$ are precisely the semi-invariants with respect to the character $\chi^u : T \rightarrow \mathbb{K}^*$.

Now we recall some background around good quotients; general references are [26] and [6]. A *good quotient* for a T -invariant open set $U \subset Z$ is an affine, T -invariant morphism $\pi : U \rightarrow X$ such that the pullback map $\pi^* : \mathcal{O}_X \rightarrow \pi_*(\mathcal{O}_U)^T$ to the sheaf of invariants is an isomorphism. If a T -invariant subset $U \subset Z$ admits a good quotient, then the quotient space is denoted by $U // T$, and we will refer to $U \subset Z$ as a *good T -set*.

It is a basic property of a good quotient $\pi : U \rightarrow U // T$ that each of its fibres $\pi^{-1}(x)$ contains precisely one closed T -orbit, and this orbit lies in the closure of any other T -orbit of $\pi^{-1}(x)$. From this one may derive the universal property: any T -invariant morphism $U \rightarrow Y$ factors uniquely through $U \rightarrow U // T$. This, by the way, justifies the notation $U \rightarrow U // T$. One writes $U \rightarrow U/T$ for a good quotient, if it is *geometric*, i.e., its (set-theoretical) fibres are precisely the T -orbits.

In the study of good T -sets the following concept is useful, compare [6]: an inclusion $U \subset U'$ of good T -sets in Z is said to be *T -saturated* if U is a full inverse image under the quotient map $U' \rightarrow U' // T$. Due to the basic property of good quotients just mentioned, the set U is T -saturated in U' if and only if any closed T -orbit of U is also closed in U' .

Let us define the good T -sets $U \subset Z$ we are looking for. First recall that an *A_2 -variety* is a variety X such that any two $x, x' \in X$ admit a common affine neighborhood in X . For example, any quasiprojective variety is an A_2 -variety. It is shown in [32] that the normal A_2 -varieties are precisely those admitting closed embeddings into toric varieties.

Definition 1.1. (Compare [31].) We say that a good T -set $U \subset T$ is a *$(T, 2)$ -set* if the quotient space $U // T$ is an A_2 -variety. By a *$(T, 2)$ -maximal* subset of Z we mean a $(T, 2)$ -set that does not occur as a proper T -saturated subset of some other $(T, 2)$ -set.

Our aim is a combinatorial description of all $(T, 2)$ -maximal subsets $U \subset Z$. Let us introduce the necessary data.

As in [4], we define the *orbit cone* associated to $z \in Z$ to be the (convex, polyhedral) cone $\omega(z) \subset K_{\mathbb{Q}}$ generated by all $u \in K$ that admit an element $f \in R_u$ with $f(z) \neq 0$. There are only finitely many orbit cones, and we have

$$\omega(z) = \omega(Z) := \operatorname{cone}(u \in K; R_u \neq 0)$$

for all points z of a nonempty open subset of Z . Moreover, for any point $z \in Z$, the toric variety $\text{Spec}(\mathbb{K}[\omega(z) \cap K])$ is the normalization of its T -orbit closure $C_Z(T \cdot z)$ in Z . In particular, we have a bijection

$$\{T\text{-orbits in } C_Z(T \cdot z)\} \rightarrow \text{faces}(\omega(z)), \quad T \cdot z' \mapsto \omega(z').$$

Definition 1.2. Let $\Omega(Z)$ denote the collection of all orbit cones $\omega(z)$, where $z \in Z$.

- (i) By a *2-connected collection* we mean a subcollection $\Psi \subset \Omega(Z)$ such that $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$ holds for any two $\tau_1, \tau_2 \in \Psi$.
- (ii) By a *2-maximal collection*, we mean a 2-connected collection, which is not a proper subcollection of any other 2-connected collection.
- (iii) We say that a 2-connected collection Ψ is a *face* of a 2-connected collection Ψ' (written $\Psi \preceq \Psi'$), if for any $\omega' \in \Psi'$ there is an $\omega \in \Psi$ with $\omega \preceq \omega'$.

Note that the 2-maximal collections form a partially ordered set with respect to the face relation defined above. Here comes the link to the torus action.

Definition 1.3. To any collection $\Psi \subset \Omega(Z)$, we associate a T -invariant subset $U(\Psi) \subset Z$ as follows:

$$U(\Psi) := \{z \in Z; \omega_0 \preceq \omega(z) \text{ for some } \omega_0 \in \Psi\}.$$

We state two basic observations. The first one shows that the set $U(\Psi)$ associated to a 2-maximal collection is a union of certain localizations, and thus is in particular open in Z . The second one characterizes the closed T -orbits in $U(\Psi)$.

Lemma 1.4. Let Ψ be a 2-maximal collection in $\Omega(Z)$. Then any $z \in U(\Psi)$ admits an open neighborhood $U(z) \subset U(\Psi)$ such that for every $u \in \omega(z)^\circ$ there is an $n > 0$ and a homogeneous $f \in R_{nu}$ with $U(z) = Z_f$.

Proof. Choose homogeneous $h_1, \dots, h_r \in R$ such that $h_i(z) \neq 0$ holds and the orbit cone $\omega(z)$ is generated by $\deg(h_i)$, where $1 \leq i \leq r$. For $w \in \mathbb{Z}_{>0}^r$, consider

$$f^w := h_1^{w_1} \dots h_r^{w_r} \in R.$$

Then the sets Z_{f^w} do not depend on the particular choice of $w \in \mathbb{Z}_{>0}^r$. Moreover for any $u \in \omega(z)^\circ$, we find some w with

$$\deg(f^w) = w_1 \deg(h_1) + \dots + w_r \deg(h_r) \in \mathbb{Q}_{>0} u.$$

In order to see that $U(z) = Z_{f^w}$ is as wanted, we still have to verify that any $z' \in Z_{f^w}$ belongs to $U(\Psi)$. By construction, we have $\omega(z) \subset \omega(z')$. Consider $\omega_0 \in \Psi$ with $\omega_0 \preceq \omega(z)$. Then ω_0° is contained in the relative interior of some face $\omega'_0 \preceq \omega(z')$. By maximality of Ψ , we have $\omega'_0 \in \Psi$, and hence $z' \in U(\Psi)$. \square

Lemma 1.5. *Let Ψ be a 2-connected collection in $\Omega(Z)$, and let $z \in U(\Psi)$. Then the orbit $T \cdot z$ is closed in $U(\Psi)$ if and only if $\omega(z) \in \Psi$ holds.*

Proof. First let $T \cdot z$ be closed in $U(\Psi)$. By the definition of $U(\Psi)$, we have $\omega_0 \preceq \omega(z)$ for some $\omega_0 \in \Psi$. Consider the closure $C_Z(T \cdot z)$ of $T \cdot z$ taken in Z , and choose $z_0 \in C_Z(T \cdot z)$ with $\omega(z_0) = \omega_0$. Again by the definition of $U(\Psi)$, we have $z_0 \in U(\Psi)$. Since $T \cdot z$ is closed in $U(\Psi)$, we obtain $z_0 \in T \cdot z$, and hence $\omega = \omega_0 \in \Psi$.

Now, let $\omega(z) \in \Psi$. We have to show that any $z_0 \in C_Z(T \cdot z) \cap U(\Psi)$ lies in $T \cdot z$. Clearly, $z_0 \in C_Z(T \cdot z)$ implies $\omega(z_0) \preceq \omega(z)$. By the definition of $U(\Psi)$, we have $\omega_0 \preceq \omega(z_0)$ for some $\omega_0 \in \Psi$. Since Ψ is a 2-connected collection, we have $\omega_0^\circ \cap \omega(z)^\circ \neq \emptyset$. Together with $\omega_0 \preceq \omega(z)$ this implies $\omega_0 = \omega(z_0) = \omega(z)$, and hence $z_0 \in T \cdot z$. \square

A first major step towards the main result of this section is to show that the 2-maximal collections define $(T, 2)$ -sets.

Proposition 1.6. *For any 2-maximal collection Ψ in $\Omega(Z)$, the associated $U(\Psi)$ is a $(T, 2)$ -set.*

Proof. We regard $U(\Psi)$ as a union of sets $U(z)$ as provided in Lemma 1.4, where $z \in U(\Psi)$ runs through those points that have a closed T -orbit in $U(\Psi)$; according to Lemma 1.5 these are precisely the points $z \in U(\Psi)$ with $\omega(z) \in \Psi$.

First consider two such $z_1, z_2 \in U(\Psi)$. Then we have $\omega(z_i) \in \Psi$, and we can choose homogeneous $f_1, f_2 \in R$ such that $\deg(f_1) = \deg(f_2)$ lies in $\omega(z_1)^\circ \cap \omega(z_2)^\circ$ and $U(z_i) = Z_{f_i}$ holds. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} Z_{f_1} & \longleftarrow & Z_{f_1 f_2} & \longrightarrow & Z_{f_2} \\ \downarrow \parallel T & & \downarrow \parallel T & & \downarrow \parallel T \\ X_{f_1} & \longleftarrow & X_{f_1 f_2} & \longrightarrow & X_{f_2} \end{array}$$

where the upper horizontal maps are open embeddings, the downwards maps are good quotients for the respective affine T -varieties, and the lower horizontal arrows indicate the induced morphisms of the affine quotient spaces.

By the choice of f_1 and f_2 , the quotient f_2/f_1 is an invariant function on Z_{f_1} , and the inclusion $Z_{f_1 f_2} \subset Z_{f_1}$ is just the localization by f_2/f_1 . Since f_2/f_1 is invariant, the latter holds as well for the quotient spaces; that means that the map $X_{f_1 f_2} \rightarrow X_{f_1}$ is localization by f_2/f_1 .

Now, cover $U(\Psi)$ by sets $U(z_i)$ with $T \cdot z$ closed in $U(\Psi)$. The preceding consideration allows gluing of the maps $U(z_i) \rightarrow U(z_i) \parallel T$ along $U_{ij} \rightarrow U_{ij} \parallel T$, where $U_{ij} := U(z_i) \cap U(z_j)$. This gives a good quotient $U(\Psi) \rightarrow U(\Psi) \parallel T$. The quotient space is separated, because we always have surjective multiplication maps

$$\mathcal{O}(Z_{f_i})^T \otimes \mathcal{O}(Z_{f_j})^T \rightarrow \mathcal{O}(Z_{f_i f_j})^T.$$

In order to see that $X = U(\Psi) // T$ is even an A_2 -variety, consider $x_1, x_2 \in X$. Then there are f_i as above with $x_i \in X_{f_i}$. The union $X_{f_1} \cup X_{f_2}$ is quasiprojective, because, for example, the set $X_{f_1} \setminus X_{f_2}$ defines an ample divisor. It follows that there is a common affine neighborhood of x_1, x_2 in $X_{f_1} \cup X_{f_2}$ and hence in X . \square

Now we discuss an inverse construction, associating to any T -invariant open subset a collection of orbit cones.

Definition 1.7. To any T -invariant open subset $U \subset Z$, we associate a set of orbit cones, namely

$$\Psi(U) := \{\omega(z); z \in U \text{ with } T \cdot z \text{ closed in } U\}.$$

The following statement shows that, when starting with a $(T, 2)$ -set, we obtain a 2-connected collection. Its proof is the only place, where factoriality of the ring R comes in; in fact, it would even be sufficient to require that for any Weil divisor on $Z = \text{Spec}(R)$ some positive multiple is principal.

Proposition 1.8. *Suppose that $Z = \text{Spec}(R)$ is factorial. Then, for any $(T, 2)$ -set $U \subset Z$, the associated set $\Psi(U)$ is a 2-connected collection.*

Proof. By definition, the elements of $\Psi(U)$ are precisely the orbit cones $\omega(z)$, where $T \cdot z$ is a closed subset of U . We have to show that for any two cones $\omega(z_i) \in \Psi(U)$, their relative interiors intersect nontrivially.

Consider the quotient $\pi: U \rightarrow U // T$, and let $V \subset U // T$ be a common affine neighborhood of $\pi(z_1)$ and $\pi(z_2)$. Since R is factorial, there is a homogeneous function $f \in R$ vanishing precisely on the complement $Z \setminus \pi^{-1}(V)$. It follows that the degree of f lies in the relative interior of both cones, $\omega(z_1)$ and $\omega(z_2)$. \square

We are ready to formulate the main result of this section; it gives a complete description of the $(T, 2)$ -maximal sets $U \subset Z$, and describes the possible inclusions of such sets.

Theorem 1.9. *Let the algebraic torus $T = \text{Spec}(\mathbb{K}[K])$ act on a factorial variety $Z = \text{Spec}(R)$. Then we have mutually inverse bijections of finite sets:*

$$\begin{aligned} \{2\text{-maximal collections in } \Omega(Z)\} &\leftrightarrow \{(T, 2)\text{-maximal subsets of } Z\}, \\ \Psi &\mapsto U(\Psi), \\ \Psi(U) &\leftarrow \Psi. \end{aligned}$$

These bijections are order-reversing maps of partially ordered sets in the sense that we always have

$$\Psi \preceq \Psi' \quad \Leftrightarrow \quad U(\Psi) \supseteq U(\Psi').$$

Corollary 1.10. (Świąćicka [31]) *The number of $(T, 2)$ -maximal subsets $U \subset Z$ is finite.*

Remark 1.11. Theorem 1.9 shows that parts of Białynicki-Birula's program [6] can be carried out to obtain open subsets with good quotient for the action of a connected reductive group G on Z : fix any maximal torus $T \subset G$, determine the $(T, 2)$ -maximal open subset $U(\Psi)$ along the lines of Theorem 1.9, and then [16, Theorem 1.1] provides good G -sets:

$$W(\Psi) = \bigcap_{g \in G} g \cdot U(\Psi).$$

Proof of Theorem 1.9. So far, we know from Proposition 1.6 that $U(\Psi)$ is a $(T, 2)$ -set, and from Proposition 1.8 that $\Psi(U)$ is a 2-connected collection. We begin with two auxiliary statements:

Claim 1. *For any 2-maximal collection Ψ in $\Omega(Z)$ we have $\Psi(U(\Psi)) = \Psi$.*

Consider any $\omega \in \Psi(U(\Psi))$. By the definition of $\Psi(U(\Psi))$, we have $\omega = \omega(z)$ for some $z \in U(\Psi)$ such that $T \cdot z$ is closed in $U(\Psi)$. According to Lemma 1.5, the latter implies $\omega \in \Psi$. Conversely, let $\omega \in \Psi$. Then we have $z \in U(\Psi)$, for any $z \in Z$ with $\omega(z) = \omega$. Moreover, Lemma 1.5 tells us that $T \cdot z$ is closed in $U(\Psi)$. This implies $\omega \in \Psi(U(\Psi))$.

Claim 2. *Let $U \subset Z$ be a $(T, 2)$ -set, and let Ψ be any 2-maximal collection in $\Omega(Z)$ with $\Psi(U) \subset \Psi$. Then we have a T -saturated inclusion $U \subset U(\Psi)$.*

First let us check that U is in fact a subset of $U(\Psi)$. Given $z \in U$, we may choose $z_0 \in C_Z(T \cdot z)$ such that $T \cdot z_0$ is closed in U . By definition of $\Psi(U)$, we have $\omega(z_0) \in \Psi(U)$, and hence $\omega(z_0) \in \Psi$. Thus, $\omega(z_0) \preceq \omega(z)$ implies $z \in U(\Psi)$.

In order to see that the inclusion $U \subset U(\Psi)$ is T -saturated, let $z \in U$ with $T \cdot z$ closed in U . We have to show that any $z_0 \in C_Z(T \cdot z)$ with $T \cdot z_0$ closed in $U(\Psi)$ belongs to $T \cdot z$. On the one hand, given such z_0 , we obtain, using Claim 1:

$$\omega(z_0) \in \Psi(U(\Psi)) = \Psi.$$

On the other hand, the definition of $\Psi(U)$ yields $\omega(z) \in \Psi$, and $z_0 \in C_Z(T \cdot z)$ implies $\omega(z_0) \preceq \omega(z)$. Since Ψ is a 2-connected collection, the relative interiors of $\omega(z_0)$ and $\omega(z)$ intersect nontrivially, and we obtain $\omega(z_0) = \omega(z)$. This gives $z_0 \in T \cdot z$.

Now we turn to the assertions of the theorem. First we show that the assignment $\Psi \mapsto U(\Psi)$ is well defined, i.e., that $U(\Psi)$ is $(T, 2)$ -maximal. Consider any T -saturated inclusion $U(\Psi) \subset U$ with a $(T, 2)$ -set $U \subset Z$. Using Claim 1, we obtain

$$\Psi = \Psi(U(\Psi)) \subset \Psi(U).$$

By maximality of Ψ , this implies $\Psi = \Psi(U)$. Thus, we obtain $U(\Psi) = U(\Psi(U))$. By Claim 2, the latter set comprises U , and thus, we see $U(\Psi) = U$. In other words, $U(\Psi)$ is $(T, 2)$ -maximal.

Thus, we have a well-defined map $\Psi \rightarrow U(\Psi)$ from the 2-maximal collections in $\Omega(Z)$ to the $(T, 2)$ -maximal subsets of Z . According to Claim 1, this map is injective. To see surjectivity, consider any $(T, 2)$ -maximal $U \subset Z$. Choose a 2-maximal collection Ψ with $\Psi(U) \subset \Psi$. Claim 2 then shows $U = U(\Psi)$. The fact that $\Psi \mapsto U(\Psi)$ and $U \mapsto \Psi(U)$ are inverse to each other is then obvious.

Finally, let us turn to the second statement of the assertion. The subset $U(\Psi')$ is contained in $U(\Psi)$ if and only if any closed T -orbit in $U(\Psi')$ is contained in $U(\Psi)$. By Lemma 1.5, the points with closed T -orbit in $U(\Psi')$ are precisely the points $z \in Z$ with $\omega(z) \in \Psi'$. By the definition of $U(\Psi)$, such a point z belongs to $U(\Psi)$ if and only if $\omega(z)$ has a face contained in Ψ . \square

Finally, let us ask for good T -sets $U \subset Z$ with a projective quotient space $U // T$. Clearly, such good T -sets are $(T, 2)$ -maximal, and we would like to see to which class of 2-maximal collections they correspond.

For this purposes, it is reasonable to assume that $R_0 = \mathbb{K}$ holds, i.e., that there are only constant invariants. Then the good T -sets $U \subset Z$ with a projective quotient space are precisely the sets of semistable points of the T -linearizations of the trivial line bundle over Z , compare [22, Converse 1.12 and 1.13].

In [4], the following description of the collection of the possible sets of semistable points is given: for any $u \in K$ define its *GIT-cone* to be the (convex, polyhedral) cone

$$\kappa(u) := \bigcap_{u \in \omega(z)} \omega(z).$$

The main result of [4, Section 2] says that these GIT-cones form a fan, and that they are in a (well defined) order reversing one-to-one correspondence to the possible sets of semistable points via

$$\kappa \mapsto U(\kappa) := \bigcup_{f \in R_{nu}, n > 0} Z_f, \quad \text{where } u \in \kappa^\circ.$$

So, given a GIT-cone $\kappa \subset K_{\mathbb{Q}}$, one may fix any $u \in K$ belonging to the relative interior of $\kappa^\circ \subset \kappa$, and then $U(\kappa)$ is the union of all localizations Z_f , where $f \in R$ is homogeneous of some degree nu with $n > 0$.

Proposition 1.12. *In the setting of Theorem 1.9, suppose that $R_0 = \mathbb{K}$ holds. Then there is an order preserving injection of finite sets:*

$$\begin{aligned} \{\text{GIT-cones}\} &\rightarrow \{2\text{-maximal collections in } \Omega(Z)\}, \\ \kappa &\mapsto \Psi_\kappa := \{\omega \in \Omega(Z); \kappa^\circ \subset \omega^\circ\}. \end{aligned}$$

The resulting open sets $U(\Psi_\kappa) \subset Z$ are precisely the good T -sets in Z that have a projective quotient space.

Proof. First recall that every good T -set $U \subset Z$ with a projective quotient space $U // T$ is $(T, 2)$ -maximal. Thus, our task is to show that for any of these $U \subset Z$ we have $\Psi(U) = \Psi_\kappa$, with a unique GIT-cone κ .

Given a good T -set $U \subset Z$ with a projective quotient space, we know that it is a set of semistable points, i.e., we have $U = U(\kappa)$ with a unique GIT-cone κ . Consider any $u \in \kappa^\circ$. Then we have

$$U = U(\kappa) = \bigcup_{f \in R_{nu}, n > 0} Z_f = \{z \in Z; u \in \omega(z)\}.$$

From the last description we infer that the closed T -orbits of $U = U(\kappa)$ are precisely those $T \cdot z \subset Z$, for which we have $u \in \omega(z)^\circ$. This implies $\Psi(U) = \Psi_\kappa$, and we are done. \square

2. Small V -embeddings

In this section, we prepare our study of equivariant embeddings of homogeneous spaces with small boundary. We provide a framework for comparing varieties having a prescribed finitely generated total coordinate ring. But first, we recall the latter notion and a little background, compare [3] and [14].

Consider a normal variety X with free finitely generated divisor class group, and choose a subgroup $K \subset \text{WDiv}(X)$ of the group of Weil divisors such that the canonical map $K \rightarrow \text{Cl}(X)$ is an isomorphism. The corresponding *total coordinate ring* $\mathcal{R}_K(X)$ is defined as the algebra of global sections of a certain K -graded sheaf:

$$\mathcal{R}_K(X) := \Gamma(X, \mathcal{R}_K), \quad \text{where } \mathcal{R}_K := \bigoplus_{D \in K} \mathcal{O}(D).$$

Note that for any homogeneous element $f \in \mathcal{R}_K(X)$ of degree $D \in K$ we obtain the sections over $X \setminus Z(f)$, where $Z(f)$ is the support of $\text{div}(f) + D$, as

$$\Gamma(X \setminus Z(f), \mathcal{R}_K) = \mathcal{R}_K(X)_f.$$

This shows in particular that \mathcal{R}_K is locally of finite type, if $\mathcal{R}_K(X)$ is finitely generated. Clearly, \mathcal{R}_K is locally of finite type for any locally factorial X .

If \mathcal{R}_K is locally of finite type, then we may construct a (generalized) universal torsor: consider the relative spectrum $\hat{X} := \text{Spec}_X(\mathcal{R}_K)$. The K -grading of \mathcal{R}_K defines an action of the Neron–Severi torus $T := \text{Spec}(\mathbb{K}[K])$ on \hat{X} , and the canonical map $\hat{X} \rightarrow X$ is a good quotient for this action.

The variety \hat{X} is quasiffine. If $\mathcal{R}_K(X)$ is finitely generated, then, setting $\bar{X} := \text{Spec}(\mathcal{R}_K(X))$, and denoting by $F(X) \subset \mathcal{R}_K(X)$ the collection of homogeneous global sections such that $X \setminus Z(f)$ is affine, we obtain \hat{X} as a union of localizations in \bar{X} , compare [4, Proposition 3.1] and [3, Proof of 4.2 a)]:

$$\hat{X} = \bigcup_{f \in F(X)} \bar{X}_f \subset \bar{X}.$$

Now we are ready to begin our investigation of varieties X with prescribed finitely generated total coordinate ring R . We start with an open subvariety $W \subset \operatorname{Spec}(R)$ that will serve as the universal torsor of a “common” open subvariety V of the varieties X .

More precisely, let W be a quas affine variety with trivial divisor class group $\operatorname{Cl}(W)$ such that $R := \mathcal{O}(W)$ satisfies $R^* = \mathbb{K}^*$. Moreover, suppose that there is a free action of an algebraic torus $T = \operatorname{Spec}(\mathbb{K}[K])$ on W admitting a geometric quotient $q: W \rightarrow V$. This action determines a grading

$$R = \bigoplus_{u \in K} R_u.$$

We are now going to fix a group of divisors $K_V \subset \operatorname{WDiv}(V)$ on the orbit space. By freeness of the action on W , we may fix a lattice basis $\{u_1, \dots, u_k\}$ of K such that for each u_i there is a nonzero rational function $f_i \in Q(R)$, which is homogeneous of degree u_i . We set

$$f_u := f_1^{n_1} \dots f_k^{n_k}, \quad \text{for } u = n_1 u_1 + \dots + n_k u_k.$$

Using once more freeness of the T -action on W , we may cover W by T -invariant open subsets $W_\alpha \subset W$ such that for every $i = 1, \dots, k$ there are invertible elements $\eta_{i,\alpha} \in \mathcal{O}(W_\alpha)_{u_i}$. Similarly as before, we define

$$\eta_{u,\alpha} := \eta_{1,\alpha}^{n_1} \dots \eta_{k,\alpha}^{n_k}, \quad \text{for } u = n_1 u_1 + \dots + n_k u_k.$$

This allows us to associate to any degree $u \in K$ a Cartier divisors D_u on the orbit space $V = W/T$. Namely, denoting $V_\alpha := q(W_\alpha)$, we define this divisor via local equations:

$$D_u|_{V_\alpha} := \operatorname{div}\left(\frac{f_u}{\eta_{u,\alpha}}\right).$$

We shall denote by $K_V \subset \operatorname{WDiv}(V)$ the group of divisors formed by the D_u , where $u \in K$. We list basic properties of this construction, showing in particular that W is a universal torsor for the orbit space $V = W/T$:

Proposition 2.1. *The natural maps $K \rightarrow K_V$ and $K_V \rightarrow \operatorname{Cl}(V)$ are isomorphisms. Moreover, we have a canonical isomorphism of sheaves, given over open $V_0 \subset V$ by*

$$q_* \mathcal{O}_W \rightarrow \mathcal{R}_{K_V}, \quad \Gamma(V_0, q_* \mathcal{O}_W)_u \ni h \mapsto \frac{h}{f_u} \in \Gamma(V_0, \mathcal{R}_{K_V})_u.$$

This isomorphism of sheaves induces a canonical equivariant isomorphism $\hat{V} \rightarrow W$ of quas affine T -varieties.

Proof. In order to see that $K \mapsto K_V$ is an isomorphism, we only have to care about injectivity. This follows, for example, from $q^*(D_u) = \operatorname{div}(f_u)$ and $R^* = \mathbb{K}^*$. The fact that $K_V \rightarrow \operatorname{Cl}(V)$ is an isomorphism is proven as in [3, Lemma 5.1], using standard arguments

on computing the Picard group of a good quotient space provided in [19]. The remaining statements are obvious. \square

From now on, we assume that the \mathbb{K} -algebra $R = \mathcal{O}(W)$ is finitely generated. Then there is a canonical affine closure $Z := \operatorname{Spec}(R)$. The complement $Z \setminus W$ is of codimension at least two in Z , and Z is a factorial variety. Moreover, the T -action on W extends uniquely to a T -action on Z .

We consider *small V -embeddings*, i.e., open embeddings $\iota: V \rightarrow X$ into a normal variety X such that $X \setminus \iota(V)$ is of codimension at least two in X . By a morphism of small V -embeddings $\iota_1: V \rightarrow X_1$ and $\iota_2: V \rightarrow X_2$, we mean a morphism $\varphi: X_1 \rightarrow X_2$ such that the following triangle is commutative:

$$\begin{array}{ccc} & V & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ X_1 & \xrightarrow{\varphi} & X_2 \end{array}$$

The following observation shows that small embeddings arise in a functorial way as quotient spaces of saturated extensions of the good T -set $W \subset Z$.

Proposition 2.2. *Let $U \subset U' \subset Z$ be good T -sets, both containing W as a T -saturated subset. Then we have:*

- (i) *the induced map $V \rightarrow U // T$ of quotient spaces is a small V -embedding;*
- (ii) *the induced map $U // T \rightarrow U' // T$ is a morphism of V -embeddings.*

Proof. Since $W \subset U$ is a T -saturated inclusion, the induced map of quotients $V \rightarrow U // T$ is an open embedding. Moreover, this embedding must be small, since $W \subset U$ is so. The second statement is obvious. \square

If $\iota: V \rightarrow X$ is a small V -embedding, then any divisor $D \in K_V$ extends, by closing the support of $\iota_*(D)$, to a Weil divisor on X . Denoting by $K_X \subset \operatorname{WDiv}(X)$ the group of divisors obtained this way, we have canonical isomorphisms

$$\operatorname{Cl}(V) \cong K_V \cong K_X \cong \operatorname{Cl}(X).$$

The open embedding $\iota: V \rightarrow X$ induces a canonical isomorphism of graded sheaves $\iota^* \mathcal{R}_{K_X} \rightarrow \mathcal{R}_{K_V}$, and hence we have an open embedding $\hat{\iota}: \hat{V} \rightarrow \hat{X}$. Moreover, the canonical isomorphism

$$R \rightarrow \Gamma(X, \mathcal{R}_{K_X}), \quad R_u \ni h \mapsto (\iota^*)^{-1} \left(\frac{h}{f_u} \right) \in \Gamma(X, \mathcal{R}_{K_X})_u$$

defines an isomorphism $\bar{X} \rightarrow Z$. The image of the open subset $\hat{X} \subset \bar{X}$ is an open subset $W_X \subset Z$. As we will see now, the construction sending X to W_X is inverse to the one given in Proposition 2.2.

Proposition 2.3. *For every small embedding $\iota: V \rightarrow X$, we have a T -saturated inclusion $W \subset W_X$, and a commutative diagram*

$$\begin{array}{ccc} & Z & \\ \swarrow \subset & & \searrow \subset \\ W & \xrightarrow{\quad} & W_X \\ \downarrow /T & & \downarrow //T \\ V & \xrightarrow{\quad \iota \quad} & X \end{array}$$

Moreover, given any T -saturated extension $W \subset U$ with a good T -set $U \subset Z$, we have $W_U // T = U$.

Proof. Note that the isomorphism $\hat{V} \rightarrow W$ given in Proposition 2.1 uniquely extends to an isomorphism $\bar{V} \rightarrow Z$. Thus, by construction, we have the following commutative diagram of T -equivariant maps:

$$\begin{array}{ccc} & Z & \\ \cong \nearrow & & \nwarrow \cong \\ \bar{V} & \xrightarrow{\quad \bar{\iota} \quad} & \bar{X} \\ \subset \uparrow & & \uparrow \subset \\ \hat{V} & \xrightarrow{\quad \hat{\iota} \quad} & \hat{X} \end{array}$$

This gives the commutative diagram in the assertion. The fact that $W \subset W_X$ is T -saturated, follows from the fact that this obviously holds for $\hat{\iota}(\hat{V}) \subset \hat{X}$.

Finally, let $W \subset U$ be a T -saturated extension. Set $X := U // T$, and consider the corresponding small embedding $V \rightarrow X$. We have to show that $W_X = U$ holds. Consider the commutative diagram

$$\begin{array}{ccccc} U & \leftarrow & W & \rightarrow & W_X \\ & \searrow & \downarrow & \swarrow & \\ & & V & & \\ q_U \swarrow & & \downarrow & & \searrow q_X \\ & & X & & \end{array}$$

Let $X_0 \subset X$ be any affine open subset. Then its boundary $D := X \setminus X_0$ is of pure codimension one in X . Since q_U as well as q_X are affine, and all sets of the upper row have small boundary in Z , we obtain

$$q_U^{-1}(X_0) = Z \setminus \overline{q^{-1}(\iota^{-1}(D))} = q_X^{-1}(X_0),$$

where $\iota: V \rightarrow X$ denotes the embedding, and $q: W \rightarrow V$ is the geometric quotient fixed before. The assertion now follows by covering X with affine open subsets $X_0 \subset X$. \square

Next we note that morphisms of small embeddings correspond to inclusions of T -saturated extensions of W in Z .

Proposition 2.4. *Any morphism $\varphi: X_1 \rightarrow X_2$ of small V -embeddings $\iota_1: V \rightarrow X_1$ and $\iota_2: V \rightarrow X_2$ gives rise to a commutative diagram*

$$\begin{array}{ccc} & W & \\ \swarrow & \downarrow & \searrow \\ W_{X_1} & \xrightarrow{\quad} & W_{X_2} \\ \downarrow & \downarrow & \downarrow \\ X_1 & \xrightarrow[\varphi]{} & X_2 \end{array}$$

$\iota_1 \quad \quad \quad \iota_2$

The map $W_{X_1} \rightarrow W_{X_2}$ is open inclusion. Moreover, $\varphi: X_1 \rightarrow X_2$ is an open embedding if and only if $W_{X_1} \subset W_{X_2}$ is T -saturated.

Proof. The morphism $\varphi: X_1 \rightarrow X_2$ gives rise to a pullback homomorphism of sheaves $\varphi^*\mathcal{R}_{X_2} \rightarrow \mathcal{R}_{X_1}$, which in turn defines a commutative diagram

$$\begin{array}{ccc} & \hat{V} & \\ \hat{\iota}_2 \swarrow & & \searrow \hat{\iota}_1 \\ \hat{X}_1 & \xrightarrow[\hat{\varphi}]{} & \hat{X}_2 \end{array}$$

Applying the canonical embeddings $\hat{V} \rightarrow Z$, and $\hat{X}_i \rightarrow Z$, we obtain the commutative diagram of the assertion.

The fact that the morphism $\varphi: X_1 \rightarrow X_2$ lifts to an inclusion is due to the fact that ι_1 and ι_2 do so. Moreover, the fact that open embeddings $\varphi: X_1 \rightarrow X_2$ correspond to T -saturated inclusions $W_{X_1} \subset W_{X_2}$, follows from the observations that $W_{X_i} \rightarrow X_i$ is a good quotient and that open embeddings $\varphi: X_1 \rightarrow X_2$ lift to T -saturated open embeddings $\hat{\varphi}: \hat{X}_1 \rightarrow \hat{X}_2$. \square

The preceding three propositions may be summarized as follows: on the one hand, we have the category of saturated W -extensions, that means good T -sets $U \subset Z$ containing W as a T -saturated subset, together with inclusions as morphisms; on the other hand, we have the category of small V -embeddings. We showed:

Corollary 2.5. *The assignments $U \mapsto U \parallel T$ and $X \mapsto W_X$ define equivalences between the categories of saturated W -extensions and small V -embeddings, and they are essentially inverse to each other.*

As in [3], we say that an A_2 -variety X is A_2 -maximal if for any open embedding $X \rightarrow X'$ into an A_2 -variety X' such that the complement $X' \setminus X$ is of codimension at least two, we already have $X = X'$.

Corollary 2.6. *Let $\iota : V \rightarrow X$ be a small V -embedding into an A_2 -variety X . Then X is A_2 -maximal if and only if $W_X \subset Z$ is $(T, 2)$ -maximal.*

As there exist complete normal varieties which are not A_2 -varieties, see [31, Example 6.4], we would like to have a similar statement comprising also the complete case. Generalizing completeness, we introduce the following concept:

Definition 2.7. We say that a variety X is 2-complete if it admits no open embeddings $X \subset X'$ with $X' \setminus X$ nonempty and of codimension at least two.

For small V -embeddings, also 2-completeness may be expressed in terms of quotients. Recall from [6] that a good T -set $U \subset Z$ is said to be T -maximal if it is maximal with respect to T -saturated inclusion in the collection of all good T -sets of Z .

Corollary 2.8. *Let $\iota : V \rightarrow X$ be a small V -embedding. Then X is 2-complete if and only if $W_X \subset Z$ is T -maximal.*

3. Small equivariant embeddings

In this section, we present the main results of the paper. We study small equivariant embeddings $G/H \subset X$, where $H \subset G$ is a “Grosshans extension”; the precise definition is given below. Among other things, we obtain finiteness of the numbers of isomorphism classes of 2-complete and A_2 -maximal small equivariant G/H -embeddings, and we give a combinatorial description in the latter case.

Let us fix the setup. Throughout this section, we denote by G a connected linear algebraic group having a trivial character group $\mathbb{X}(G)$, and trivial Picard group $\text{Pic}(G)$. For example, G might be any connected simply connected semisimple group, like the special linear group $\text{SL}(n, \mathbb{K})$.

Let $H \subset G$ be a closed subgroup. As usual, we mean by an *equivariant embedding* of the homogeneous space G/H an irreducible, normal G -variety X together with a base point $x_0 \in X$ such that H equals the isotropy group G_{x_0} of $x_0 \in X$ and the morphism

$$G/H \rightarrow X, \quad gH \mapsto g \cdot x_0$$

is a (G -equivariant) open embedding. A *morphism* of two equivariant embeddings X and X' of G/H is a G -equivariant morphism $X \rightarrow X'$ sending the base point $x_0 \in X$ to the base point $x'_0 \in X'$. Note that if a morphism of G/H -embeddings exists, then it is unique.

Definition 3.1. By a *small equivariant G/H -embedding* we mean an equivariant G/H -embedding with a normal variety X such that the complement $X \setminus G \cdot x_0$ is of codimension at least two in X .

Let us recall from [15] the necessary concepts from algebraic group theory. The subgroup $H \subset G$ is said to be *observable* if G/H is a quasiaffine variety. Moreover, $H \subset G$ is called a *Grosshans subgroup* if it is observable and the algebra of global functions $\mathcal{O}(G/H) = \mathcal{O}(G)^H$ is finitely generated.

Remark 3.2. In each of the following cases, the subgroup $H \subset G$ is a Grosshans subgroup:

- G/H is quasiaffine and spherical or of complexity one, see [23];
- H is the unipotent radical of a parabolic subgroup of G , see [15];
- H is the generic stabilizer of a factorial affine G -variety, see [15].

The property of being a Grosshans subgroup can (tautologically) be characterized in terms of small embeddings; more precisely, we observe the following.

Remark 3.3. The subgroup $H \subset G$ is Grosshans if and only if there is a small embedding $G/H \rightarrow X$ into a (normal) affine variety X . In this case, X is the spectrum of $\mathcal{O}(G/H)$.

We are ready to introduce the notion of a “Grosshans extension.” Let $K := \mathbb{X}(H)$ denote the character group of the subgroup $H \subset G$. Then we have an associated diagonalizable group $T := \text{Spec}(\mathbb{K}[K])$, a canonical epimorphism $\pi : H \rightarrow T$, and we may consider its kernel:

$$H_1 := \ker(\pi) = \bigcap_{\chi \in K} \ker(\chi) \subset H.$$

Definition 3.4. We say that $H \subset G$ is a *Grosshans extension*, if H is connected, and $H_1 \subset G$ is a Grosshans subgroup.

We now present our results; the proofs are given at the end of the section. The first observation is a characterization of Grosshans extensions in the spirit of Remark 3.3.

Proposition 3.5. *A connected closed subgroup $H \subset G$ is a Grosshans extension if and only if there is a small embedding $G/H \rightarrow X$ into a normal variety X with finitely generated free divisor class group and finitely generated total coordinate ring.*

The next statement shows that, in many cases, for a Grosshans extension $H \subset G$, small embeddings $G/H \rightarrow X$ are automatically equivariant.

Proposition 3.6. *Let $H \subset G$ be a Grosshans extension, and let $\iota : G/H \rightarrow X$ be a small embedding. If X is 2-complete or A_2 -maximal, then the canonical G -action on $\iota(G/H) \subset X$ extends to the whole X .*

We come to the main results. The first one is the following finiteness statement on small equivariant embeddings.

Theorem 3.7. *Let G be a connected linear algebraic group with trivial Picard group and only trivial characters, and let $H \subset G$ be a Grosshans extension.*

- (i) *The number of isomorphism classes of 2-complete small equivariant G/H -embeddings is finite.*
- (ii) *The number of isomorphism classes of A_2 -maximal small equivariant G/H -embeddings is finite.*

As a direct application, we note the following statement on the group of G -equivariant automorphisms:

Corollary 3.8. *In the setting of Theorem 3.7, let $N_G(H)$ be the normalizer of H in G , and $N^0 \subset N_G(H)/H$ the unit component. Then, for any 2-complete or A_2 -maximal small equivariant G/H -embedding X , the group $\text{Aut}_G(X)$ of G -equivariant automorphisms contains N^0 as its unit component.*

Now, fix a Grosshans extension $H \subset G$. Note that $K = \mathbb{X}(H)$ is a lattice, and thus $T = \text{Spec}(\mathbb{K}[K])$ is a torus. Consider the subgroup $H_1 \subset H$ as defined before. Then $W := G/H_1$ is a quas affine variety, and it comes with canonical actions of G and $T \cong H/H_1$, given by

$$g \cdot (g'H_1) := gg'H_1, \quad (hH_1) \cdot g'H_1 := g'h^{-1}H_1.$$

The algebra $R := \mathcal{O}(W)$ is finitely generated, and we have a canonical affine closure $Z := \text{Spec}(R)$. The actions of G and T both extend to Z , and, obviously, they commute. The T -action on Z defines a grading of the algebra of functions:

$$R := \bigoplus_{\mu \in K} R_\mu.$$

Having in mind this grading, we may speak, as in the first section, about the collection $\Omega(Z)$ of all T -orbit cones $\omega(z) \subset K_{\mathbb{Q}}$, and also about the GIT-cones $\kappa \subset K_{\mathbb{Q}}$. We will work with the following notions.

Definition 3.9. We say that a subset $\Psi \subset \Omega(Z)$ is an *interior collection* if it contains the weight cone $\omega(Z)$. By an *interior GIT-cone* we mean a GIT-cone $\kappa \subset K_{\mathbb{Q}}$ with $\kappa^\circ \subset \omega(Z)^\circ$.

Our main result is a description of the category of A_2 -maximal equivariant small G/H -embeddings of a given Grosshans extension $H \subset G$ in terms of interior 2-maximal collections $\Psi \subset \Omega(Z)$. In the case $\mathcal{O}(G/H) = \mathbb{K}$, it comprises also a description of all projective equivariant small G/H -embeddings.

Together with the face relations “ \preceq ” as morphisms, the interior 2-maximal collections $\Psi \subset \Omega(Z)$ form a category. Moreover, we may associate to any Ψ the variety $U(\Psi) // T$,

and this assignment is functorial: if we have $\Psi \preceq \Psi'$, then there is an induced morphism of the quotient spaces

$$U(\Psi') // T \rightarrow U(\Psi) // T.$$

As we will see, the set $U(\Psi) \subset Z$ is G -invariant, and thus the G -action descends to the quotient space $U(\Psi) // T$. Moreover this space comes with a canonical base point, namely $\pi(e_G H_1)$, where $\pi : U(\Psi) \rightarrow U(\Psi) // T$ denotes the quotient map.

Theorem 3.10. *Let G be a connected linear algebraic group with trivial Picard group and only trivial characters, and let $H \subset G$ be a Grosshans extension. Then we have a contravariant equivalence of categories:*

$$\{\text{interior 2-maximal collections}\} \rightarrow \left\{ \begin{array}{l} A_2\text{-maximal small equivariant} \\ G/H\text{-embeddings} \end{array} \right\},$$

$$\Psi \mapsto U(\Psi) // T.$$

If moreover $\mathcal{O}(G/H) = \mathbb{K}$ holds, then we have in addition a contravariant equivalence of categories:

$$\{\text{interior GIT-cones}\} \rightarrow \left\{ \begin{array}{l} \text{projective small equivariant} \\ G/H\text{-embeddings} \end{array} \right\},$$

$$\kappa \mapsto U(\kappa) // T.$$

The condition $\mathcal{O}(G/H) = \mathbb{K}$ has been studied by several authors; for example, various characterizations of this property and concrete examples can be found in [8,9], and [15, Section 23 B].

Remark 3.11. If we drop the assumption $\mathcal{O}(G/H) = \mathbb{K}$ in the second part of Theorem 3.10, then the interior GIT-cones correspond to those A_2 -maximal small equivariant G/H -embeddings, which are in addition quasiprojective.

Using Proposition 1.12, we observe that in the case of a small character group $\mathbb{X}(H)$ and $\mathcal{O}(G/H) = \mathbb{K}$, every A_2 -maximal small G/H -embeddings is projective. More precisely, we obtain the following.

Corollary 3.12. *For a Grosshans extension $H \subset G$ with $\mathbb{X}(H)$ of rank at most two and $\mathcal{O}(G/H) = \mathbb{K}$, every A_2 -maximal small equivariant G/H -embedding is projective.*

Let us begin to prove the results. A first basic step is the following group theoretical observation.

Proposition 3.13. *Let G be any linear algebraic group, and let $H \subset G$ be a closed subgroup. Then the following subgroup is observable:*

$$H_1 = \bigcap_{\chi \in \mathbb{X}(H)} \ker(\chi).$$

Moreover, if the subgroup $H \subset G$ is connected, then the above group H_1 has only trivial characters.

Proof. By Chevalley's theorem, there exist a rational finite-dimensional G -module V , and a nonzero vector $v \in V$ such that H is the stabilizer of the line $\mathbb{K}v$. Let $\chi_0 \in \mathbb{X}(H)$ such that $h \cdot v = \chi_0(h)v$ for any $h \in H$. Then $H_0 := \ker(\chi_0)$ equals the isotropy subgroup G_v of v , and thus is observable in G , see [15, Theorem 2.1]. Note that we have

$$H_1 = \bigcap_{\chi \in \mathbb{X}(H)} \ker(\chi|_{H_0}) \subset H_0.$$

Since the intersection of observable subgroups is again observable, it suffices to show that each $H_\chi := \ker(\chi|_{H_0})$ is observable. For this, use again [15, Theorem 2.1] to realize the one-dimensional H_0 -module given by $\chi|_{H_0}$ as an H_0 -submodule $\mathbb{K}v_\chi$ of a G -module V_χ . Then H_χ is the isotropy group of (v, v_χ) in the G -module $V \oplus V_\chi$, and hence it is observable.

To see the second assertion, let $H = LR_u(H)$ be the Levi decomposition, where $R_u(H)$ is the unipotent radical and L is connected reductive, compare [24, Section 6.4]. The subgroup L locally splits into the direct product $L = T^c L^s$, where T^c is the central torus and L^s is a semisimple subgroup coinciding with the commutator subgroup of L , compare [24, Section 4.1.3]. Clearly, $R_u(H)$ and L^s are contained in H_1 . On the other hand, $H/(L^s R_u(H))$ is isomorphic to $T^c/(T^c \cap L^s)$, and hence is a torus. This implies $H_1 = L^s R_u(H)$, which proves $\mathbb{X}(H_1) = 0$. \square

The next step is to ensure that we are in the setup of the preceding two sections. Let $H \subset G$ be any connected subgroup, set $W := G/H_1$ and $V := G/H$.

Lemma 3.14. *The variety W is quasiaffine, factorial, and satisfies $\mathcal{O}^*(W) = \mathbb{K}^*$. Moreover, the T -action on W is free, and the canonical map $W \rightarrow V$ is a geometric quotient for this action. In particular, $W \rightarrow V$ is a universal torsor.*

Proof. Proposition 3.13 tells us that $H_1 \subset G$ is a Grosshans subgroup, and hence $W = G/H_1$ is quasiaffine. It is obvious that $T \cong H/H_1$ acts freely on $W = G/H_1$. Thus, the canonical map $W \rightarrow V$ must be a geometric quotient for the T -action on W . Moreover $\mathcal{O}^*(W) = \mathbb{K}^*$ follows from $\mathcal{O}^*(G) = \mathbb{K}^*$, which in turn is due to $\mathbb{X}(G) = 0$, see, for example, [19, Proposition 1.2].

It remains to show that W is factorial, i.e., has trivial divisor class group $\text{Cl}(W)$. Since W is smooth, we have $\text{Cl}(W) = \text{Pic}(W)$. The latter group occurs in the exact sequence

$$\mathbb{X}(G) \longrightarrow \mathbb{X}(H_1) \longrightarrow \text{Pic}(W) \longrightarrow \text{Pic}(G),$$

see, for example, [19, Proposition 3.2]. We assumed $\mathbb{X}(G) = 0$ and $\text{Pic}(G) = 0$. Thus $\text{Pic}(W) = 0$ follows from $\mathbb{X}(H_1) = 0$. Proposition 2.1 then tells us that $W \rightarrow V$ is a universal torsor. \square

Proof of Proposition 3.5. The “only if” part follows directly from Lemma 3.14: it tells us that $G/H_1 \rightarrow G/H$ is a universal torsor and hence the trivial embedding $G/H \rightarrow G/H$ is as wanted.

Conversely, if there is a small embedding $G/H \rightarrow X$ as in the assertion, then G/H has a free finitely generated divisor class group and a finitely generated total coordinate ring $\mathcal{R}(G/H)$. Again Lemma 3.14 shows that $G/H_1 \rightarrow G/H$ is a universal torsor. Thus $\mathcal{O}(G/H_1) \cong \mathcal{R}(G/H)$ holds. In particular, this algebra is finitely generated, which means that H_1 is a Grosshans subgroup. \square

From now on, $H \subset G$ is a Grosshans extension, and, thus $R = \mathcal{O}(W)$ is finitely generated. Note that then $Z = \text{Spec}(R)$ contains W as an open subset with small complement $Z \setminus W$. In particular, we have $\text{Cl}(Z) = 0$, which means that Z is factorial.

Lemma 3.15. *Let $\Psi \subset \Omega(Z)$ be a 2-maximal collection of orbit cones. Then the associated set $U(\Psi) \subset Z$ is G -invariant.*

Proof. Since the actions of T and G on Z commute, we see that for any $z \in Z$ and any $g \in G$, we have $\omega(z) = \omega(g \cdot z)$. The assertion thus follows from the definition of $U(\Psi)$. \square

Proof of Proposition 3.6. Let us first consider the A_2 -maximal case. According to Corollary 2.6, the small embedding $\iota: G/H \rightarrow X$, defines a $(T, 2)$ -maximal open subset $W_X \subset Z$ such that everything fits into a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\subset} & W_X \\ \downarrow /T & & \downarrow //T \\ G/H & \xrightarrow{\iota} & X \end{array}$$

where $W \subset W_X$ is a T -saturated inclusion. By Theorem 1.9, we have $W_X = U(\Psi)$ with some 2-maximal collection $\Psi \subset \Omega(Z)$. By Lemma 3.15, the set W_X is G -invariant. Thus, the action of G on W_X descends to the desired G -action on X .

If X is 2-complete, then the arguments are similar. Again, by Proposition 2.3, we have a T -saturated inclusion $W \subset W_X$, where $W_X \subset Z$ is a good T -set. Moreover, Corollary 2.8 tells us that W_X is T -maximal. Thus, [30, Corollary 2.3] yields that W_X is G -invariant, and, again, the G -action descends to the desired action on the variety X . \square

Proof of Theorem 3.7. By Proposition 3.6, the category of 2-complete small equivariant G/H -embeddings as well as the category of A_2 -maximal small equivariant G/H -embeddings are full subcategories of the category of small (not necessarily equivariant) V -embeddings, where $V = G/H$.

Thus, according to Corollaries 2.5, 2.6, and 2.8 we only need to know that the collection of T -maximal open subsets $U \subset Z$ and the collection of $(T, 2)$ -maximal open subsets $U' \subset Z$ are finite. In the first case this follows from the main result of [5], in the second case this is Corollary 1.10. \square

Proof of Corollary 3.8. First note that $N_G(H)/H$ may be identified with the group $\text{Aut}_G(G/H)$; in fact, $N_G(H)/H$ acts on G/H via

$$nH \cdot gH := gn^{-1}H.$$

Consequently, for any G -variety X with an open G -orbit isomorphic to G/H , the group $\text{Aut}_G(X)$ is a subgroup of $N_G(H)/H$.

Moreover, the group $N_G(H)/H$ acts on the set of isomorphism classes of G/H -embeddings via

$$nH \cdot (X, x_0) := (X, n^{-1} \cdot x_0).$$

Two pairs (X, x_0) and $(X, n^{-1} \cdot x_0)$ are isomorphic as G/H -embeddings if and only if $nH \in \text{Aut}_G(X)$ holds.

For 2-complete and as well A_2 -maximal equivariant G/H -embeddings X , Theorem 3.7 tells us that the respective numbers of isomorphism classes are finite. Hence, for a given X , the group $\text{Aut}_G(X)$ acts on the set of H -fixed points in G/H with finitely many orbits. This action is precisely the action of $\text{Aut}_G(X)$ on the group $N_G(H)/H$ by right multiplication, and thus, by dimension reasons, $\text{Aut}_G(X)$ contains N^0 as its unit component. \square

Proof of Theorem 3.10. Let us first check that the assignment is well defined. By Lemma 3.15, the sets $U(\Psi) \subset Z$ defined by 2-maximal collections $\Psi \subset \Omega(Z)$ are G -invariant. Moreover, any interior 2-maximal $\Psi \subset \Omega(Z)$ contains the generic orbit cone $\omega(e_G H_1)$. Consequently, $W \subset U(\Psi)$ holds, and, by Lemma 1.5, this is a T -saturated inclusion. Proposition 2.2 thus provides a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\subset} & U(\Psi) \\ \downarrow /T & & \downarrow \pi // T \\ G/H & \longrightarrow & X \end{array}$$

where $X := U(\Psi) // T$, and the induced map of quotient spaces is an open embedding. The G -action on $U(\Psi)$ descends to an action on the quotient variety X , making it into a small equivariant G/H -embedding with base point $\pi(e_G H_1)$. By Corollary 2.6 the variety X is A_2 -maximal. Hence, the assignment $\Psi \mapsto U(\Psi) // T$ is well defined.

According to Proposition 3.6, the category of A_2 -maximal small equivariant G/H -embeddings and that of A_2 -maximal small G/H -embeddings are isomorphic via sending the G -variety (X, x_0) to the embedding $G/H \rightarrow X$, $gH \mapsto g \cdot x_0$. Thus, Theorem 1.9,

together with Corollaries 2.5 and 2.6 shows that $\Psi \mapsto U(\Psi) // T$ defines a (contravariant) fully faithful functor.

Similarly as in the proof of Proposition 3.6, we show now that our functor is essentially surjective. Let X be any A_2 -maximal small equivariant G/H -embedding. Then, according to Corollary 2.6, we have a commutative diagram of T -equivariant maps with a $(T, 2)$ -maximal subset $W_X \subset Z$ and a T -saturated inclusion $W \subset W_X$:

$$\begin{array}{ccc} W & \xrightarrow{\subset} & W_X \\ \downarrow /T & & \downarrow //T \\ V & \longrightarrow & X \end{array}$$

By Theorem 1.9, we have $W_X = U(\Psi)$ for a 2-maximal collection of orbit cones $\Psi \subset \Omega(Z)$. Moreover, by Lemma 1.5, the generic orbit cone $\omega(e_G H_1)$ belongs to Ψ . Hence Ψ is an interior 2-maximal collection. Clearly, there is an induced isomorphism $X \rightarrow U(\Psi) // T$ of small equivariant G/H -embeddings.

The statement concerning the projective case is an immediate consequence of Proposition 1.12 and the A_2 -maximal case, which we just settled. \square

4. Constructing examples

The aim of this section is to provide a concrete combinatorial recipe to construct examples of small equivariant G/H -embeddings with Grosshans extensions $H \subset G$. First, due to our main results, we obviously have the following general recipe.

Construction 4.1. Let G be a connected linear algebraic group with trivial Picard group and only trivial characters. Every Grosshans extension in G arises from the following procedure:

- Take a connected Grosshans subgroup $F \subset G$ with $\mathbb{X}(F) = 0$, consider the normalizer $N_G(F)$, and the projection $\pi : N_G(F) \rightarrow N_G(F)/F$.
- Choose a maximal torus $T_F \subset N_G(F)/F$, and a surjection $Q : \mathbb{X}(T_F) \rightarrow K$ of lattices, and let $T \subset T_F$ be the corresponding subtorus.

Then $H_T := \pi^{-1}(T)$ is a Grosshans extension in G with $(H_T)_1 = F$. The small equivariant G/H_T -embeddings arise from the following procedure:

- Determine the set $\Omega(Z)$ of orbit cones of the T -action on the factorial affine variety $Z := \text{Spec}(\mathcal{O}(G)^F)$.
- Fix a 2-maximal collection $\Psi \subset \Omega(Z)$ of T -orbit cones; e.g., a collection $\Psi = \Psi_\kappa$ arising from a GIT-cone $\kappa \subset K_\mathbb{Q}$.

Then the 2-maximal collection Ψ defines an open subset $U(\Psi) \subset Z$, and the quotient $U(\Psi) // T$ is a small equivariant G/H_T -embedding.

So, the starting point of this construction is the choice of a connected Grosshans subgroup $F \subset G$; recall that Remark 3.2 gives a list of examples. The second part surely requires a certain knowledge of the algebra $\mathcal{O}(G)^F$; we refer to [1, Theorem 2] for a detailed study in the case of G being semisimple and F being the unipotent radical of a parabolic subgroup $P \subset G$.

We now introduce a class of Grosshans subgroups $F \subset G$, the extensions $H_T \subset G$ of which allow a purely combinatorial construction of small equivariant G/H_T -embeddings.

Definition 4.2. We call a connected Grosshans subgroup $F \subset G$ *suitable* if there is a system $\{f_1, \dots, f_r\} \subset \mathcal{O}(G)^F$ of T_F -homogeneous prime generators such that every cone in $\mathbb{X}_{\mathbb{Q}}(T_F)$ spanned by some of the weights of the f_i is an orbit cone of the T_F -action on Z .

Here comes the concrete recipe for the construction of small equivariant embeddings when starting with a suitable Grosshans extension:

Construction 4.3. Let $F \subset G$ be a suitable Grosshans subgroup, fix a maximal torus $T_F \subset N_G(F)/F$, and a system $\{f_1, \dots, f_r\} \subset \mathcal{O}(G)^F$ of T_F -homogeneous prime generators as in Definition 4.2.

- Choose a surjection $Q: \mathbb{X}(T_F) \rightarrow K$ of lattices, and let $T \subset T_F$ be the corresponding subtorus.
- Determine the images $u_i := Q(\deg(f_i))$. Then the set $\Omega(Z)$ of T -orbit cones consists of all cone(u_{i_1}, \dots, u_{i_p}), where $\{i_1, \dots, i_p\} \subset \{1, \dots, r\}$.
- Fix a 2-maximal collection $\Psi \subset \Omega(Z)$ of T -orbit cones; e.g., a collection $\Psi = \Psi_{\kappa}$ arising from a GIT-cone $\kappa \subset K_{\mathbb{Q}}$.

Then the 2-maximal collection Ψ defines an open subset $U(\Psi) \subset Z$, and the quotient $U(\Psi) // T$ is a small equivariant G/H_T -embedding.

In order to show that this construction really leads to concrete examples, we now present some classes of suitable Grosshans subgroups. We begin with an example providing spherical varieties.

Proposition 4.4. Let G be semisimple simply connected and $F \subset G$ be a maximal unipotent subgroup. Then F is suitable in G .

Proof. Let $\gamma_1, \dots, \gamma_s$ be fundamental weights of G with respect to a Borel subgroup $B = T_F F$, and $V(\gamma_1), \dots, V(\gamma_s)$ be corresponding simple G -modules with highest vectors $v_{\gamma_i} \in V(\gamma_i)$. Then [15, Theorem 5.4] tells us that

$$Z = \overline{G(v_{\gamma_1}, \dots, v_{\gamma_s})} \subset V(\gamma_1) \oplus \dots \oplus V(\gamma_s),$$

and the maximal torus $T_F \cong N_G(F)/F$ acts on the variety $Z = \text{Spec}(\mathcal{O}(G)^F)$ by means of

$$t \cdot (v_1, \dots, v_s) = (\gamma_1(t^{-1})v_1, \dots, \gamma_s(t^{-1})v_s).$$

For any subset $J \subseteq \{1, \dots, s\}$ consider the G -orbit through a generic point

$$v_J \in \bigoplus_{j \in J} V(\gamma_j)^F \subset V(\gamma_1) \oplus \dots \oplus V(\gamma_s).$$

The orbit $G \cdot v_J$ is contained in Z and this implies that F is suitable in G . \square

The following class of examples may produce homogeneous spaces of arbitrary high complexity. In particular, they cannot be treated by spherical methods, and hence, we go a little bit more into detail as before.

Proposition 4.5. *Let $G := \mathrm{SL}(m)$ act diagonally on $(\mathbb{K}^m)^s$, where $s \leq m - 1$, and consider the isotropy subgroup*

$$F := G_{(e_1, \dots, e_s)} = \left\{ \begin{bmatrix} E_s & A \\ 0 & B \end{bmatrix}; B \in \mathrm{SL}(m-s), A \in \mathrm{Mat}(s \times (m-s)) \right\}.$$

Then F is a connected Grosshans subgroup of G , and a possible maximal torus $T_F \subset N_G(F)/F$ is the isomorphic image of

$$T'_F := \{\mathrm{diag}(t_1, \dots, t_s, t^{-1}, 1, \dots, 1); t_i \in \mathbb{K}^*, t = t_1 \dots t_s\} \subset N_G(F).$$

Moreover, we have $Z = \mathrm{Spec}(\mathcal{O}(G)^F) = (\mathbb{K}^m)^s$, and the torus T_F acts on the variety Z via

$$t \cdot (v_1, \dots, v_s) = (t_1^{-1}v_1, \dots, t_s^{-1}v_s).$$

In particular, every cone generated by weights of the coordinate functions is a T_F -orbit cone, and thus F is suitable in G .

Proof. The complement of the open G -orbit in $(\mathbb{K}^m)^s$ is the variety of collections of linearly dependent vectors, thus it has codimension ≥ 2 . This implies

$$Z = \mathrm{Spec}(\mathcal{O}(G/F)) \cong (\mathbb{K}^m)^s.$$

In particular, F is a Grosshans subgroup of G . The normalizer $N_G(F)$ coincides with the maximal parabolic subgroup

$$P = \left\{ \begin{bmatrix} C & A \\ 0 & B \end{bmatrix} \right\} \subset \mathrm{SL}(m),$$

and we have $N_G(F)/F \cong \mathrm{GL}(s)$. Clearly, the projection π maps T'_F isomorphically onto a maximal torus of $\mathrm{GL}(s)$. The further statements are obvious. \square

In the setting of Proposition 4.5, Construction 4.3 produces small equivariant G/H -embeddings that come with the structure of a toric variety: the action of the torus \mathbb{T}^{ms} on

$(\mathbb{K}^m)^s$ commutes with that of T , and hence descends to the varieties $U(\Psi) // T$, where Ψ a 2-maximal collection, making them into toric varieties.

So it is natural to ask which further small equivariant G/H -embeddings have additionally a structure of a toric variety. First of all, using [3, Corollary 4.5], Corollaries 2.6 and 2.8 and the fact that any T -maximal subset of a linear torus actions on \mathbb{K}^n is automatically a $(T, 2)$ -set, we obtain the following.

Remark 4.6. For a toric variety X with free divisor class group and $\mathcal{O}^*(X) = \mathbb{K}^*$, the following statements are equivalent.

- (i) The variety X is 2-complete.
- (ii) The variety X is A_2 -maximal.
- (iii) The fan of X cannot be enlarged without adding new rays.

We show now that besides the examples produced via Proposition 4.5 and Construction 4.3, there is only a very limited list of 2-complete small equivariant G/H -embeddings X , where G is simple and X is toric.

Proposition 4.7. *Let G be a simple simply connected linear algebraic group. Then the 2-complete small equivariant G/H -embeddings X , where $H \subset G$ is a connected subgroup, and X admits the structure of a toric variety, arise via Construction 4.3 from the following list:*

- (i) $G = \mathrm{SL}(m)$ and $Z = (\mathbb{K}^m)^s$ with the diagonal G -action and $F \subset G$ etc. as in Proposition 4.5, or the dual G -module Z^* with the analogous data $F \subset G$ etc.,
- (ii) the group G and the G -modules Z and their duals Z^* , where G and Z are as listed below

$$\begin{array}{ll}
 G = \mathrm{SL}(2m+1), & Z = \bigwedge^2 \mathbb{K}^{2m+1}, \\
 G = \mathrm{SL}(2m+1), & Z = \bigwedge^2 \mathbb{K}^{2m+1} \times \bigwedge^2 \mathbb{K}^{2m+1}, \\
 G = \mathrm{SL}(2m+1), & Z = \bigwedge^2 \mathbb{K}^{2m+1} \times (\mathbb{K}^{2m+1})^*, \\
 G = \mathrm{Sp}(2m), & Z = \mathbb{K}^{2m}, \\
 G = \mathrm{Spin}(10), & Z = \mathbb{K}^{16},
 \end{array}$$

and $F = H_1$ is the stabilizer of a generic point in Z (Z^*), and the system of generators $\{f_1, \dots, f_r\}$ may be taken as the set of coordinate functions.

Proof. Note that $\mathrm{Cl}(X) \cong \mathrm{Cl}(G/H)$ is free, and by [12], the toric variety X has a polynomial ring as total coordinate ring. Hence X may be obtained as a good T -quotient of an open subset $U \subset \mathbb{K}^l$. By our assumption, U is a $(T, 2)$ -maximal subset of \mathbb{K}^l and thus, by Theorem 3.10, corresponds to an interior 2-maximal collection Ψ of T -orbit cones.

Moreover, we infer from Proposition 3.5 and Lemma 3.14 that H is a Grosshans extension, and that the G -action on X lifts to a prehomogeneous G -action on \mathbb{K}^l commuting with the T -action. The $(G \times T)$ -action on \mathbb{K}^l is linearizable, see [18, Proposition 5.1] for

details. Hence V is a direct sum of simple $(G \times T)$ -modules, and the T -action on V is given as in Construction 4.3.

Thus, to conclude the proof, we have to say what are the possible prehomogeneous G -modules. The list of them was obtained in [29]. \square

Let us take a quick look at a concrete example. Assume that $H \subset G$ is a Grosshans extension such that G/H is quas affine. If $\mathbb{X}(H) = 0$, then Theorem 3.10 implies that $Z = \operatorname{Spec}(\mathcal{O}(G/H))$ is the only A_2 -maximal small G/H -embedding. If H has nontrivial characters, this need no longer hold, as we shall see now.

Example 4.8. In the notation of Proposition 4.5, take $m = 3$ and $s = 2$. So, we have $G = \operatorname{SL}(3)$ acting diagonally on $(\mathbb{K}^3)^2$. Consider the subtorus

$$T' := \operatorname{diag}(t, t^{-1}, 1) \subset T'_F \subset N_G(F),$$

and set $T = \pi(T')$, where, as before, $\pi : N_G(F) \rightarrow N_G(F)/F$ is the projection. Then the corresponding Grosshans extension is

$$H_T = \left\{ \begin{bmatrix} t & 0 & a \\ 0 & t^{-1} & b \\ 0 & 0 & 1 \end{bmatrix}; t \in \mathbb{K}^*, a, b \in \mathbb{K} \right\}.$$

The algebra $\mathcal{O}(G)^F = \mathcal{O}(Z)$ is generated by the coordinate functions of $Z = (\mathbb{K}^3)^2$, and hence the weights of the generators in $\mathbb{Z} = \mathbb{X}(T)$ are $u_1 = 1$ and $u_2 = -1$. The collection of orbit cones and the possible 2-maximal collections are given by

$$\Omega(Z) = \{\mathbb{Q}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}, 0\}, \quad \Psi_0 = \{\mathbb{Q}, 0\}, \quad \Psi_1 = \{\mathbb{Q}, \mathbb{Q}_{\geq 0}\}, \quad \Psi_2 = \{\mathbb{Q}, \mathbb{Q}_{\leq 0}\}.$$

Thus, we see that for the homogeneous space G/H_T there are, up to isomorphism, precisely three A_2 -maximal small equivariant G/H_T -embeddings. We will discuss them below a little more in detail.

The set $U(\Psi_0)$ associated to Ψ_0 is the whole $Z = (\mathbb{K}^3)^2$. The resulting small equivariant G/H_T -embedding $X_0 = U(\Psi_0) // T$ is an affine cone with apex $x_1 \in X_0$; it may be realized in the G -module $\mathbb{K}^3 \otimes \mathbb{K}^3$ as the closure of the G -orbit through $(e_1 \otimes e_2)$ with the quotient map

$$U(\Psi_0) \rightarrow X_0, \quad (v_1, v_2) \mapsto v_1 \otimes v_2.$$

For the collection Ψ_1 , one has $U(\Psi_1) = \{(v_1, v_2); v_1 \neq 0\}$. The resulting small equivariant G/H_T -embedding $X_1 = U(\Psi_1) // T$ is quasiprojective but not affine. Indeed, the quotient map may be realized via

$$U(\Psi_1) \rightarrow (\mathbb{K}^3 \otimes \mathbb{K}^3) \times \mathbb{P}^2, \quad (v_1, v_2) \mapsto (v_1 \otimes v_2, \langle v_1 \rangle).$$

From Theorem 3.10 we know that there is a morphism $X_1 \rightarrow X_0$ of equivariant G/H_T -embeddings. In fact, this is the projection to $\mathbb{K}^3 \otimes \mathbb{K}^3$; this map is an isomorphism over $X_0 \setminus \{x_1\}$, and the fibre over the apex x_1 is isomorphic to \mathbb{P}^2 .

The variety $X_2 = U(\Psi_2) // T$ is isomorphic to X_1 as a G -variety, but not as a G/H_T -embedding (there is no base point preserving equivariant morphism). We may realize X_2 by the same construction as X_1 but twisted by the automorphism θ of $Z = \mathbb{K}^3 \oplus \mathbb{K}^3$, given by $\theta(v_1, v_2) = (v_2, v_1)$.

In order to see that our construction also may produce nontoric examples, look at the following case.

Proposition 4.9. *Let \mathbb{K}^{2m} be the symplectic vector space with the skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$, given as*

$$\begin{bmatrix} 0 & E_m \\ -E_m & 0 \end{bmatrix},$$

and $G = \mathrm{Sp}(2m)$ be the symplectic group. Consider the diagonal G -action on $(\mathbb{K}^{2m})^s$, where $s \leq m$, and the isotropy group

$$F := G_{(e_1, \dots, e_s)}.$$

Then F is a connected Grosshans subgroup of G , and a possible maximal torus $T_F \subset N_G(F)/F$ is the isomorphic image of

$$T'_F := \{\mathrm{diag}(t_1, \dots, t_s, 1, \dots, 1, t_1^{-1}, \dots, t_s^{-1}, 1, \dots, 1, \dots); t_i \in \mathbb{K}^*\} \subset N_G(F).$$

The affine variety $Z = \mathrm{Spec}(\mathcal{O}(G)^F)$ can be realized as the G -orbit closure of (e_1, \dots, e_s) and is given by

$$Z = \{(v_1, \dots, v_s); \langle v_i, v_j \rangle = 0 \forall i, j\}.$$

The action of T_F on the variety Z is given as

$$t \cdot (v_1, \dots, v_s) = (t_1^{-1}v_1, \dots, t_s^{-1}v_s).$$

Every cone generated by weights of the restricted coordinate functions is a T_F -orbit cone, and thus F is suitable in G .

Proof. First, note that we have $Z = G \cdot (L)^s$, where $L = \langle e_1, \dots, e_m \rangle$ is a Lagrangian subspace. This shows that the complement of the open orbit $G \cdot (e_1, \dots, e_s)$ has codimension at least two in Z . Moreover, Serre's criterion of normality shows that Z is normal. This implies $\mathcal{O}(Z) = \mathcal{O}(G/F)$.

Secondly, the normalizer $N_G(F)$ is again a maximal parabolic subgroup of G , we have $N_G(F)/F \cong \mathrm{GL}(s)$, and the claim follows. \square

Remark 4.10. If we take $s = m$ in the setting of Proposition 4.9, then the subgroup F is the unipotent radical of the maximal parabolic subgroup $P \subset G$ corresponding to the long simple root, and is given by

$$F = \left\{ \begin{bmatrix} E_m & A \\ 0 & E_m \end{bmatrix}; A = A^T \right\}.$$

Remark 4.11. In Propositions 4.4, 4.5 and 4.9 the weights u_1, \dots, u_r of the coordinate functions generate a regular cone in $\mathbb{X}_{\mathbb{Q}}(T_F)$, and the T_F -orbit cones are precisely the faces of this cone.

To finish the discussion on suitable subgroups, we give an example showing that not any connected Grosshans subgroup $F \subset G$ with $\mathbb{X}(F) = 0$ is suitable.

Example 4.12. Let F be a connected semisimple subgroup of G . Then $Z = G/F$ and the only orbit cone for the T_F -action on Z is $\mathbb{X}_{\mathbb{Q}}(T_F)$. This shows that F is not suitable in G .

5. Geometric properties

In this section, we show that the language of bunched rings developed in [3], applies to A_2 -maximal small equivariant G/H -embeddings X , provided that $H \subset G$ is a Grosshans extension. This enables us to study basic geometric properties of X . For example, we obtain existence of projective small equivariant G/H -embeddings with at most \mathbb{Q} -factorial singularities, and we can easily produce homogeneous spaces G/H that do not admit any smooth small equivariant completion.

Let us briefly recall the concepts of [3]. In the sequel, R denotes a factorial, finitely generated \mathbb{K} -algebra, faithfully graded by some lattice $K \cong \mathbb{Z}^k$ such that $R^* = \mathbb{K}^*$ holds. Here, faithfully graded means that K is generated as a lattice by the degrees $w \in K$ admitting nontrivial homogeneous elements $f \in R_w$.

Moreover, $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ is a system of homogeneous pairwise nonassociated nonzero prime elements generating R as an algebra. Note that due to $R^* = \mathbb{K}^*$ such systems always exist. Since we assume the grading to be faithful, the degrees $\deg(f_i)$ generate the lattice K .

The *projected cone* $(E \xrightarrow{Q} K, \gamma)$ associated to the system of generators $\mathfrak{F} \subset R$ consists of the surjection Q of the lattices $E := \mathbb{Z}^r$ and K sending the i th canonical base vector $e_i \in \mathbb{Z}$ to the degree $\deg(f_i) \in K$, and the cone $\gamma \subset E_{\mathbb{Q}}$ generated by e_1, \dots, e_r .

Definition 5.1. Let $(E \xrightarrow{Q} K, \gamma)$ be the projected cone associated to $\mathfrak{F} \subset R$, and suppose that for each facet $\gamma_0 \preceq \gamma$, the image $Q(\gamma_0 \cap E)$ generates the lattice K .

- (i) A face $\gamma_0 \preceq \gamma$ is called an \mathfrak{F} -*face* if the product over all f_i with $e_i \in \gamma_0$ does not belong to the ideal $\sqrt{\langle f_j; e_j \notin \gamma_0 \rangle} \subset R$.
- (ii) An \mathfrak{F} -*bunch* is a nonempty collection Φ of projected \mathfrak{F} -faces with the following properties:

- a projected \mathfrak{F} -face τ belongs to Φ if and only if for each $\tau \neq \sigma \in \Phi$ we have $\emptyset \neq \tau^\circ \cap \sigma^\circ \neq \sigma^\circ$,
- for each facet $\gamma_0 \preccurlyeq \gamma$, there is a cone $\tau \in \Phi$ such that $Q(\gamma_0)^\circ \supset \tau^\circ$ holds.

If Φ is an \mathfrak{F} -bunch in the projected cone $(E \xrightarrow{Q} K, \gamma)$ associated to $\mathfrak{F} \subset R$, then the triple (R, \mathfrak{F}, Φ) is called a *bunched ring*.

Now consider the affine variety $Z := \text{Spec}(R)$, the torus $T := \text{Spec}(\mathbb{K}[K])$, and the action $T \times Z \rightarrow Z$ given by the K -grading of R . The following statements put the above definitions into a more geometric framework.

Lemma 5.2. *Let $(E \xrightarrow{Q} K, \gamma)$ be the projected cone associated to $\mathfrak{F} \subset R$. Then the following statements hold:*

- The projected \mathfrak{F} -faces are precisely the orbit cones of the T -action on Z .*
- There is a canonical injection*

$$\begin{aligned} \{\mathfrak{F}\text{-bunches}\} &\rightarrow \{2\text{-maximal collections in } \Omega(Z)\}, \\ \Phi &\mapsto \{\omega(z); z \in Z, \tau^\circ \subset \omega(z)^\circ \text{ for some } \tau \in \Phi\}. \end{aligned}$$

Proof. To prove the first statement, we first note that the defining condition of an \mathfrak{F} -face has the following geometric meaning: it says that $\gamma_0 \preccurlyeq \gamma$ is an \mathfrak{F} -face if and only if there is a point $z \in Z$ such that

$$f_i(z) \neq 0 \quad \Leftrightarrow \quad e_i \in \gamma_0. \quad (5.2.1)$$

Consider any orbit cone $\omega(z)$ where $z \in Z$. We claim that $\omega(z) = Q(\gamma_0)$ holds for the face $\gamma_0 \preccurlyeq \gamma$ defined by

$$\gamma_0 = \text{cone}(e_i; f_i(z) \neq 0). \quad (5.2.2)$$

Obviously, we have $Q(\gamma_0) \subset \omega(z)$. For the converse inclusion, consider any homogeneous $h \in R$ with $h(z) \neq 0$. Then we have a representation

$$h = \sum \alpha_v f_1^{v_1} \dots f_r^{v_r}$$

with coefficients $\alpha_v \in \mathbb{K}$. Consequently, the degree of h is a positive combination of some of the degrees of the f_i . This shows $\omega(z) \subset Q(\gamma_0)$.

Now, given any orbit cone $\omega(z)$, this cone is the image of the face $\gamma_0 \preccurlyeq \gamma$ given as in (5.2.2). Moreover, the point z satisfies (5.2.1), showing that γ_0 is an \mathfrak{F} -face. Conversely, given any \mathfrak{F} -face $\gamma_0 \preccurlyeq \gamma$, consider $z \in Z$ as in (5.2.1). Then (5.2.2) shows that $Q(\gamma_0)$ is the orbit cone of z .

The second assertion is simply due to the observation that the bunch Φ may be reconstructed from its associated 2-maximal collection Ψ by taking the set-theoretically minimal elements of Ψ . \square

Now suppose, we are in the setting of the preceding section. That means that G is a connected affine algebraic group with $\mathbb{X}(G) = 0$ and $\text{Pic}(G) = 0$, and $H \subset G$, is a Grosshans extension. Then $W = G/H_1$ is a quas affine variety with a finitely generated algebra $R := \mathcal{O}(W)$ of global functions satisfying $R^* = \mathbb{K}^*$.

Moreover, denoting by $K := \mathbb{X}(H)$ the character lattice of H , we have the canonical action of the torus $T = \text{Spec}(\mathbb{K}[K])$ on the quas affine variety W . The actions of G and T extend to the factorial affine variety $Z := \text{Spec}(R)$; in particular the T -action defines a grading:

$$R := \bigoplus_{u \in K} R_u.$$

The basic observation of this section is that, with the above data, we are in the setting of Definition 5.1. More precisely, we observe the following.

Proposition 5.3. *There is a system $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ of homogeneous pairwise nonassociated prime generators with the following properties:*

- (i) *Let $(E \xrightarrow{Q} K, \gamma)$ be the projected cone associated to $\mathfrak{F} \subset R$. Then for every facet $\gamma_0 < \gamma$, the image $Q(E \cap \gamma_0)$ generates K as a lattice.*
- (ii) *We have canonical bijections, inverse to each other:*

$$\{\mathfrak{F}\text{-bunches}\} \leftrightarrow \{\text{interior 2-maximal collections}\},$$

$$\Phi \mapsto \{\omega(z); \tau^\circ \subset \omega(z)^\circ \text{ with } \tau \in \Phi\},$$

$$\{\omega \text{ minimal in } \Psi\} \leftrightarrow \Psi.$$

If $\mathcal{O}(G/H) = \mathbb{K}$ holds, then even every system $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ of homogeneous pairwise nonassociated prime generators fulfills the above properties.

Proof. In a first step we show that any system $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ of homogeneous pairwise nonassociated prime generators satisfies condition (i). After suitably renumbering, we may assume $\gamma_0 = \text{cone}(e_1, \dots, e_{r-1})$. Consider any $z \in Z$ with $f_r(z) = 0$. Then the isotropy group $T_z \subset T$ has as its character group $K/K(z)$, where

$$\begin{aligned} K(z) &:= \langle u \in K; f(z) \neq 0 \text{ for some } f \in R_u \rangle \\ &\subset \langle \deg(f_1), \dots, \deg(f_{r-1}) \rangle. \end{aligned}$$

Since the zero set $V(Z, f_r)$ must intersect the big G -orbit $W = G/H_1$, and T acts freely on this set, we see there are points $z \in V(Z, f_r)$ with $K(z) = K$. Thus, the displayed formula shows that the image $Q(\gamma_0 \cap E)$ generates K as a lattice.

If we are in the general case, i.e., $\mathcal{O}(G/H)$ may contain nonconstant functions, then we have to provide a suitable system $\mathfrak{F} \subset R$ of generators. For this, we first take any collection of homogeneous pairwise nonassociated prime elements $\{f_1, \dots, f_r\}$ generating R . For each of these f_i , we consider the G -stable vector subspace $V_i \subset R$ generated by $G \cdot f_i$.

Since G acts with an open orbit on Z , we have $R^G = \mathbb{K}$, and hence V_i is a nontrivial G -module. Since we have $\mathbb{X}(G) = 0$, we even see that $\dim(V_i) > 1$ holds. So, there is an $f'_i = g_i \cdot f_i$, which is not proportional to f_i . Since we have $R^* = \mathbb{K}^*$ this even means that f_i and f'_i are nonassociated primes.

Adding appropriate elements f'_i , we may enlarge the initial system of generators $\{f_1, \dots, f_r\}$ such that for any i , there is a $j \neq i$ with $\deg(f_i) = \deg(f_j)$. Then it is obvious that this new complemented system satisfies

$$Q(\gamma_0) = Q(\gamma) = \omega(Z) \quad \text{for every facet } \gamma_0 \preccurlyeq \gamma. \quad (5.3.1)$$

Let us show that this property gives the second condition. By Lemma 5.2, we have a canonical injection from the \mathfrak{F} -bunches to the 2-maximal collections. Our task is to show that the image consists precisely of the interior 2-maximal collections.

Given an \mathfrak{F} -bunch Φ , condition (5.3.1) implies that $\omega(Z)$ occurs in the associated 2-maximal collection Ψ . Since $\omega(Z)$ is the generic orbit cone, Ψ is an interior collection. Conversely, given an interior 2-maximal collection, (5.3.1) yields that the corresponding collection Φ of its set-theoretically minimal cones satisfies Definition 5.1(ii), and hence is an \mathfrak{F} -bunch.

Now suppose that $\mathcal{O}(G/H) = \mathbb{K}$ holds. Then we have to show that any system $\mathfrak{F} = \{f_1, \dots, f_r\} \subset R$ of homogeneous pairwise nonassociated prime generators satisfies condition (5.3.1). First note that $\mathcal{O}(G/H) = \mathbb{K}$ implies that $Q(\gamma) = \omega(Z)$ is pointed. Thus, in order to obtain (5.3.1), we have to show that each extremal ray of $\omega(Z)$ contains at least two of the $u_i = \deg(f_i)$.

Let us verify this. Clearly, ϱ contains at least one u_i . In order to see that there must be a second one, consider any nontrivial translate $h := g \cdot f_i$. This is as well a homogeneous function of degree u_i . Since we have $\mathbb{X}(G) = 0$ the elements f_i and h are linearly independent. Thus, there must be a representation

$$h = \sum \alpha_v f_1^{v_1} \dots f_r^{v_r}$$

with coefficients $\alpha_v \in \mathbb{K}$ such that $\alpha_v \neq 0$ for at least some v admitting a $v_j \neq 0$ with $j \neq i$. Since u_i belongs to an extremal ray, we obtain $u_j \in \varrho$. This establishes condition (5.3.1) for $\{f_1, \dots, f_r\}$, from which we deduce, as before, condition (ii) of the assertion. \square

This proposition shows that for a suitable system $\mathfrak{F} = \{f_1, \dots, f_r\}$ of generators of the algebra $R = \mathcal{O}(G/H_1)$, the small equivariant G/H -embeddings are precisely the varieties

$$X(R, \mathfrak{F}, \Phi) = U(\Psi) // T$$

arising from the bunched rings (R, \mathfrak{F}, Φ) , where Ψ is the 2-maximal collection associated to the \mathfrak{F} -bunch Φ . So, we may apply the results obtained in [3] to describe geometric properties of X .

Let us briefly provide the necessary notions. Call an \mathfrak{F} -face $\gamma_0 \preccurlyeq \gamma$ *relevant* if $Q(\gamma_0)^\circ \supset \tau^\circ$ holds for some $\tau \in \Phi$, and denote by $\text{rlv}(\Phi)$ the collection of relevant \mathfrak{F} -faces. The *covering collection* of Φ is the collection $\text{cov}(\Phi) \subset \text{rlv}(\Phi)$ of set-theoretically minimal members of $\text{rlv}(\Phi)$. Here are some of the results of [3].

Theorem 5.4. *For a suitable choice of $\mathfrak{F} \subset R$, let $X := X(R, \mathfrak{F}, \Phi)$ be the small G/H -embedding arising from an \mathfrak{F} -bunch Φ . Then the following statements hold:*

- (i) *The variety X is locally factorial if and only if $Q(\gamma_0 \cap E)$ generates the lattice K for every $\gamma_0 \in \text{rlv}(\Phi)$.*
- (ii) *The variety X is \mathbb{Q} -factorial if and only if any cone of Φ is of full dimension in $K_{\mathbb{Q}}$.*
- (iii) *The rational divisor class group of X is given by $\text{Cl}_{\mathbb{Q}}(X) \cong K_{\mathbb{Q}}$, and inside $K_{\mathbb{Q}}$ the Picard group of X is given by*

$$\text{Pic}(X) = \bigcap_{\gamma_0 \in \text{cov}(\Phi)} Q(\text{lin}(\gamma_0) \cap E).$$

Moreover, inside $K_{\mathbb{Q}}$, the cones of semiample and ample divisor classes of X are given by

$$\text{SAmple}(X) = \bigcap_{\tau \in \Phi} \tau, \quad \text{Ample}(X) = \bigcap_{\tau \in \Phi} \tau^\circ.$$

Note that the semiample and the ample cone depend only on the bunch Φ , and might as well be expressed in terms of the corresponding 2-maximal collection Ψ .

As a very first application of Theorem 5.4, we give an existence statement on small equivariant G/H -embeddings in the spirit of [9, Théorème 1], but additionally guaranteeing “mild” singularities.

Corollary 5.5. *Let G be a connected linear algebraic group trivial Picard group and only trivial characters, and let $H \subset G$ be a Grosshans extension with $\mathcal{O}(G)^H = \mathbb{K}$. Then there exists a projective small equivariant G/H -embedding with at most \mathbb{Q} -factorial singularities.*

Proof. Let $R := \mathcal{O}(G)^{H_1}$, let $\mathfrak{F} \subset R$ be any system of pairwise nonassociated homogeneous prime generators, and take any \mathfrak{F} -bunch arising from a GIT-cone of full dimension. Then the corresponding small G/H -embedding is as wanted. \square

We will now apply the language of bunched rings to study the concrete examples arising from the constructions of the preceding section more in detail. As it concerns a good part of them, we first note the following.

Remark 5.6. Given a bunched ring (R, \mathfrak{F}, Φ) , where $R = \mathbb{K}[z_1, \dots, z_r]$ is a polynomial ring, and $\mathfrak{F} = \{z_1, \dots, z_r\}$ consists of the indeterminates, we are in the setting of toric varieties, and then, in addition to Theorem 5.4, there are simple combinatorial criteria for smoothness [2, Proposition 8.3] and completeness [2, Proposition 8.6].

Now we begin the study of examples. The first one gives smooth small equivariant G/H -embeddings; recall from [10, Section 5] that the existence of a smooth projective small G/H -embedding implies that G/H is generically rationally connected.

Example 5.7. In the setting of Proposition 4.5, let $m := 4$ and $s := 3$. Then the torus $T_F \subset N_G(F)/F$ is of dimension three, and it may be identified with

$$\{\text{diag}(t_1, t_2, t_3, t_1^{-1}t_2^{-1}t_3^{-1}); t_i \in \mathbb{K}^*\} \subset N_G(F).$$

We consider the projection $\mathbb{X}(T_F) \rightarrow \mathbb{Z}^2$ sending the canonical generators of the character group $\mathbb{X}(T_F)$ to the following lattice vectors

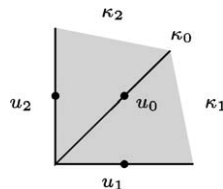
$$u_1 := (1, 0), \quad u_0 := (1, 1), \quad u_2 := (0, 1).$$

Thus, speaking more concretely, we deal with $G = \text{SL}(4)$, the Grosshans subgroup $F \subset G$ as in Proposition 4.5, a two-dimensional torus $T \subset N_G(F)/F$, and the Grosshans extension $H_T \subset G$ given by

$$H_T = \left\{ \begin{bmatrix} t_1 & 0 & 0 & a_1 \\ 0 & t_1 t_2 & 0 & a_2 \\ 0 & 0 & t_2 & a_3 \\ 0 & 0 & 0 & t_1^{-2} t_2^{-2} \end{bmatrix}; t_1, t_2 \in \mathbb{K}^*, a_i \in \mathbb{K} \right\}.$$

Let us determine the bunched rings (R, \mathfrak{F}, Φ) describing the possible A_2 -maximal small equivariant G/H_T -embeddings. First of all, $R = \mathcal{O}(G)^F$ is the ring of functions of $G/F = (\mathbb{K}^4)^3$. So, R is a polynomial ring, and as system of generators $\{f_1, \dots, f_{12}\} \subset R$, one may take the collection of indeterminates.

The remaining task is to determine the possible \mathfrak{F} -bunches. As we know from Corollary 3.12, these bunches correspond to projective varieties, and hence we only need to know the GIT-fan in $\mathbb{Z}^2 \cong \mathbb{X}(T)$ of the action of T on G/F . This in turn is easy to determine; it looks as follows:



So, the possible \mathfrak{F} -bunches are those arising from the interior GIT-cones κ_1 , κ_0 and κ_2 as indicated above, and they are explicitly given by

$$\Phi_1 = \{\kappa_1\}, \quad \Phi_0 = \{\kappa_0\}, \quad \Phi_2 = \{\kappa_2\}.$$

Let X_i denote the small equivariant G/H_T -embedding corresponding to the \mathfrak{F} -bunch Φ_i . Then, applying the results on the geometry of X_i mentioned in Theorem 5.4 and Remark 5.6, we see, for example, that X_1 and X_2 are smooth, whereas X_0 has non- \mathbb{Q} -factorial singularities.

Moreover, all varieties X_i have a divisor class group of rank two, and X_0 has a Picard group of rank one. Finally, Theorem 3.10 tells us that the possible morphisms of G/H_T -embeddings are

$$X_1 \rightarrow X_0 \leftarrow X_2.$$

By determining explicitly the varieties $U(\kappa_i)$ over X_i one may describe these morphisms explicitly, and it turns out that for $i = 1, 2$ the exceptional locus of $X_i \rightarrow X_0$ is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3$ and is contracted to a \mathbb{P}^3 lying in X_0 .

Moreover, one obtains that, as G -varieties, X_1 and X_2 are isomorphic, but, of course, not as G/H -embeddings. This shows in particular that $N_G(H)/H$ is not contained in the group of G -equivariant automorphisms of X_1 .

By slight modification of the preceding example, we present a homogeneous space $\mathrm{SL}(4)/H$ admitting equivariant completions with small boundary but no smooth ones. Existence of such examples is due to M. Brion, as mentioned in [10].

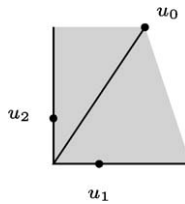
Example 5.8. In the setting of Proposition 4.5, let $m := 4$ and $s := 3$. As before, T_F is of dimension three, but now we consider the projection $\mathbb{X}(T_F) \rightarrow \mathbb{Z}^2$ sending the canonical generators to

$$u_1 := (1, 0), \quad u_0 := (2, 3), \quad u_2 := (0, 1).$$

Concretely this means that we have again the Grosshans subgroup $F \subset G = \mathrm{SL}(4)$, as in Proposition 4.5, but another two-dimensional torus $T \subset N_G(F)/F$. The Grosshans extension $H_T \subset G$ this time is given by

$$H_T = \left\{ \begin{bmatrix} t_1 & 0 & 0 & a_1 \\ 0 & t_1^2 t_2^3 & 0 & a_2 \\ 0 & 0 & t_2 & a_3 \\ 0 & 0 & 0 & t_1^{-3} t_2^{-4} \end{bmatrix} ; t_1, t_2 \in \mathbb{K}^*, a_i \in \mathbb{K} \right\}.$$

The possible small equivariant G/H -completions arise from T -maximal open subsets of $Z = (\mathbb{K}^4)^3$. All of them are toric, hence A_2 -maximal, hence projective, use, e.g., Corollary 3.12. Thus the GIT-fan for the T -action on $Z = (\mathbb{K}^4)^3$ gives the full information; it look as follows:



Using Theorem 5.4(i), we see that each of the three possible projective small equivariant G/H -embeddings is singular; in two cases, we have \mathbb{Q} -factorial singularities, in the remaining one, it is even worse.

Example 5.9. In the setting of Proposition 4.5, let $m = 7$ and $s = 6$. So, we have $G = \mathrm{SL}(7)$ acting diagonally on $(\mathbb{K}^7)^6$, and the torus T_F is of dimension six. Set $K := \mathbb{Z}^3$ and consider the map $\mathbb{X}(T_F) \rightarrow K$ sending the canonical generators to

$$e_1, e_2, e_3, \quad w_1 := e_1 + e_2, \quad w_2 := e_1 + e_3, \quad w_3 := e_2 + e_3.$$

Let $\mathfrak{F} = \{f_1, \dots, f_{42}\}$ be the indeterminates of the polynomial ring $\mathcal{O}(G)^H$. Then the following cones in $K_{\mathbb{Q}}$ define an \mathfrak{F} -bunch:

$$\mathrm{cone}(e_3, w_1, w_2), \quad \mathrm{cone}(e_1, w_1, w_3), \quad \mathrm{cone}(e_2, w_2, w_3), \quad \mathrm{cone}(w_1, w_2, w_3).$$

Combining [2, Example 11.2] and [2, Construction 11.4] shows that the corresponding small equivariant G/H_T -embedding $X(R, \mathfrak{F}, \Phi)$ is complete and \mathbb{Q} -factorial but not projective.

Finally, we give a concrete example showing that the language of bunched rings also applies in the nontoric case.

Example 5.10. In the setting of Proposition 4.9, let $m = 3$, and $s = 3$. Then the maximal torus $T_F \subset N_G(F)/F$ is of dimension three. Consider a surjection $\mathbb{X}(T_F) \rightarrow \mathbb{Z}^2$, sending the canonical generators to $u_1, u_0, u_2 \in \mathbb{Z}^2$. Then our Grosshans extension $H_T \subset \mathrm{Sp}(6)$ consists of the matrices

$$\begin{bmatrix} \chi^{u_1}(t) & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & \chi^{u_2}(t) & 0 & a_{12} & a_{22} & a_{23} \\ 0 & 0 & \chi^{u_3}(t) & a_{13} & a_{23} & a_{33} \\ 0 & 0 & 0 & \chi^{-u_1}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi^{-u_2}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \chi^{-u_3}(t) \end{bmatrix}, \quad t \in T; \quad a_{ij} \in \mathbb{K}.$$

Taking u_i as in Example 5.7, we obtain three projective small equivariant G/H_T -embeddings X_1, X_0 , and X_2 . By Theorem 5.4, the varieties X_1 and X_2 are locally factorial and X_3 is not \mathbb{Q} -factorial

In fact, an analysis of the singular locus of the cone $Z = \text{Spec}(\mathcal{O}(G)^F)$ shows that open subsets $U(\Psi_1)$ and $U(\Psi_2)$ lying over X_1 and X_2 , respectively, are smooth. Thus [3, Proposition 5.6] yields that the varieties X_1 and X_2 are even smooth.

Taking u_i as in Example 5.8, one obtains another torus T , and other projective small equivariant G/H_T -embeddings X_1 , X_0 , and X_2 . Then Theorem 5.4 tells us that X_1 and X_2 are \mathbb{Q} -factorial, but not locally factorial.

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